THE q-DEFORMED ALGEBRAS $su(n)_q$ AND THEIR APPLICATIONS

A THESIS
SUBMITTED IN PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE
OF
DOCTOR OF PHILOSOPHY IN PHYSICS
IN THE
UNIVERSITY OF CANTERBURY

by
Cindy Rae Smithies
(nee Lienert)

University of Canterbury
1995

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Since its beginning in the early 1980's, the subject of q-deformed algebras has expanded rapidly. Many of the techniques and structures of Lie algebras carry over to the q-deformations of Lie algebras. This thesis develops the Racah-Wigner algebra for q-deformations of Lie algebras and looks at some applications.

Knowing the Racah-Wigner algebra, it was clear the recursive techniques developed by Butler and others for calculating coupling coefficients and $6j$-symbols could be extended to q-deformed algebras. This allows a more general approach to finding coefficients for any q-deformation of a Lie algebra than the approaches previously known. This was the subject of a paper published in 1992 (Lienert and Butler, 1992a).

The R-matrices are special elements appearing in q-deformed algebras and form the basis of many of the applications of q-deformed algebras. The symmetries and relations of R-matrices are tied in with those of the coupling coefficients and $6j$-symbols. One of the relations satisfied by the coupling coefficients and R-matrices, the pentagonal equation, can be used as a recursion relation. The R-matrices can thus be calculated by a building up method similar to that used for the coupling coefficients. This new method allows R-matrices to be calculated far more efficiently. This work has been published (Lienert and Butler, 1992b).

One of the applications of q-deformed algebras is to finding polynomials to describe knots. Based on the Racah-Wigner algebra of q-deformed algebras the diagrammatic approach of Guadagnini (1992) has been extended. Skein relations were known for knot polynomials based on q-deformed algebras but these are insufficient in most cases for calculating the polynomial of a knot. The diagrammatic techniques based on the Racah-Wigner algebra allowed new relations to be found and hence the calculation of two classes of knot polynomials. One class is equivalent to those already known, the other class it was hoped would be more powerful, allowing more knots to be distinguished. However little difference was found. This work has been prepared for publication.

q-deformed algebras are the algebraic structure satisfied by solutions to the Yang-Baxter equation. The solutions to the Yang-Baxter equation are known as R-matrices, mentioned above. The Yang-Baxter equation is found in various guises in several areas of physics. One such appearance is as the factorization equation of 1+1 dimensional quantum field theory. Yang in 1967 gave this as the necessary equation to find solutions to a problem in many-body scattering in 1+1 dimensions (Yang, 1967). A similar equation arose in exactly solvable models of 2 dimensional statistical mechanics. Baxter was the first to find such a model and give the equation such a model satisfies (Baxter, 1972, 1982). Both the quantum field theory and statistical mechanics models are exactly solvable or integrable. Classical integrable models, such as those found in soliton theory, have long been known. A key equation in these is the classical Yang-Baxter equation which is related to the Jacobi identity of Lie algebras. The structure and representations of Lie algebras have been of much assistance in the study of integrable models. Faddeev, Sklyanin and others recognized the common equation in the work of Yang, Baxter and others and applied the techniques of the classical inverse method from integrable models to form the quantum inverse scattering method (Sklyanin *et al*, 1979; Kulish and Sklyanin, 1980). Sklyanin uncovered an algebraic structure satisfied by Baxter's known solution to the Yang-Baxter equation (Sklyanin, 1982). The structure was a one-parameter deformation of the universal enveloping algebra of the Lie algebra $su(2)$. In studying further this structure and generalizing to other algebras, new solutions of the Yang-Baxter equation without spectral parameter have been found.

The q-deformations of other algebras were defined by Drinfel'd (1985). They are 'quasi-triangular Hopf algebras'. The 'quasi-triangular' refers to their having a special element, the universal R-matrix, which satisfies the quantum Yang-Baxter equation. For ordinary Lie algebras, the R-matrix is simply the identity so that the q-deformed algebras are indeed a generalization of Lie algebras.

The q-deformed algebras have a similar representation theory to Lie algebras because of their common structure (Rosso, 1988; Lusztig, 1988). The Racah-Wigner algebra also has similarities, but there is additional structure due to the deformation parameter q. The Racah-Wigner algebra has been studied in great detail for $su(2)_q$ by Nomura and others (Nomura, 1989; Hou *et al*, 1990a; Kirillov and Reshetikhin, 1988; Vaksman, 1989). The coupling coefficients and $6j$-symbols are well known. For other groups, Reshetikhin (1987) covers some of the Racah-Wigner algebra. Very few coupling coefficients are known for other groups. Ma (1990a, 1990b) gives tables of some $su(3)_q$ coefficients.

The Racah-Wigner algebra and in particular the coupling coefficients and $6j$-symbols are important in the applications of both Lie algebras and q-deformed algebras. The algebra $su(2)_q$, like its $q = 1$ equivalent, has found application in the study of atomic and molecular structure and spectra beginning from the work of Biedenharn (1989) and Macfarlane (1989) on the q-equivalent of harmonic oscillators. Since then many systems have been studied (Bonatsos *et al*, 1990; Chang and Yan, 1991; Iwao, 1990a, 1990b; Raychev *et al*, 1990; Celeghini *et al*, 1992). A better understanding of the structure of q-deformed algebra and knowledge of coupling coefficients will assist progress in these and other applications.

The operator form of the universal R-matrix has long been known for many algebras (Burroughs, 1990; Reshetikhin, 1987). R-matrix can be expressed in matrix form by the way it acts on representations of $su(n)_q$. This explicit matrix form has only been calculated for a few groups. It is known for all representations of $su(2)_q$ (Nomura, 1989; Kirillov and Reshetikhin, 1988), for a few of $su(3)_q$ (Ma, 1990a, 1990b) and the fundamental representation of other Lie groups (Reshetikhin, 1987).

R-matrices find applications, both because they are solutions to the Yang-Baxter equation, and also because the Yang-Baxter equation without spectral parameter is related to the braid group relation. Knowing the explicit form of R-matrices can lead to new exactly solvable models. New matrix representations of braid groups can also be found. R-matrices have further properties that allow knot polynomials to be obtained. While R-matrices known from exactly solvable models give some knot polynomials, those found from the study of q-deformed algebras give a new hierarchy of knot polynomials.

The Racah-Wigner algebra of q-deformed algebras needs to be well understood in order to apply q-deformed algebras in the areas mentioned above. R-matrices need to be explicitly calculated to find solutions of the Yang-Baxter equation or to find new matrix representations for braids. We address these problems in this thesis. In addition, we study one application in detail, that of finding knot invariants.

In Chapter 2 we define the q-deformed algebras. The concept of Hopf algebras is introduced. The properties of the special element, the universal R-matrix, are given. The representation theory of q-deformed algebras is reviewed.

The Racah-Wigner algebra for $su(n)_q$ is developed in Chapter 3. Vector coupling coefficients, recoupling coefficients and R-matrices are introduced. Equations relating the coefficients are proved. A recursive method is developed for calculating each type of matrix. Those for the vector coupling coefficients and recoupling coefficients are similar to those developed for the non-deformed case by Butler and Wybourne (Butler and Wybourne, 1976; Butler, 1976). A new recursive method for calculating R-matrices is described using the pentagonal relation as a recursion relation. This work has been published in two papers (Lienert and Butler, 1992a, 1992b).

In Chapters 4 and 5 we give explicit examples of the recursive methods. The vector coupling coefficients, recoupling coefficients and R-matrices of $su(2)_q$ are calculated in Chapter 4. While all of these were previously known, the method used here is easily generalizable. This is illustrated in Chapter 5 where the primitive coupling coefficient and a large class of R-matrices are calculated for $su(3)_q$.

Chapter 6 is a review of knot theory and knot invariants. The braid group is defined and the process of obtaining knot polynomials from matrix representations of braid groups is outlined.

The process of obtaining knot invariants from the q-deformed algebras $su(n)_q$ is described in Chapter 7. Explicit calculations of invariants based on the $\{1\}$ and $\{2\}$ representations are presented and their properties discussed. Having calculated the $\{2\}$ polynomials for over 200 knots, we are able to draw some conclusions about the $\{2\}$ polynomial. The $\{2\}su(n)_q$ polynomial is better than the $\{1\}su(n)_q$ polynomial at detecting the handedness or lack of handedness of knots with no exceptions being found. However, while it distinguished all knots having the same $\{1\}su(n)_q$ polynomial, it had as many pairs.

Chapter 2

q-deformed algebras

The q-deformed algebras $su(n)_q$ are one-parameter deformations of the universal enveloping algebras of $su(n)$. They are non-commutative and non-cocommutative but, because they are quasi-triangular Hopf algebras, they have interesting properties.

In this chapter, we will explain what is meant by a quasi-triangular Hopf algebra. The q-deformed algebras $su(n)_q$ will be defined in the manner of Drinfel'd (1986) and Jimbo (1985).

2.1 Hopf algebras

A bialgebra is both an algebra having a multiplication m, and identity id, and a co-algebra having a co-multiplication Δ, co-unit ϵ analogous to a unit with the algebra and co-algebra being compatible. If there is also an antipode γ analogous to an inverse we have a Hopf algebra (Abe, 1980). The co-multiplication (or co-product) is in a sense a 'sharing out' of an element between two spaces whereas a multiplication is a merging of elements in two spaces. To be a Hopf algebra, the following relations, namely the associativity condition, antipode condition and co-unit condition, must be satisfied

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\Delta(a) &= (\Delta\otimes\mathrm{id})\Delta(a) && (2.1)\\ \mathrm{m}(\mathrm{id}\otimes\gamma)\Delta(a) &= \mathrm{m}(\gamma\otimes\mathrm{id})\Delta(a)=\epsilon(a)1 && (2.2)\\ (\epsilon\otimes\mathrm{id})\Delta(a) &= (\mathrm{id}\otimes\epsilon)\Delta(a)=a && (2.3)\end{aligned}$$

where a is an element of the algebra.

The relation $\Delta' = \sigma\circ\Delta$ where σ is a permutation, $\sigma(x\otimes y)=y\otimes x$, is another possible comultiplication for the Hopf algebra, having the same properties above but with antipode $\gamma'=\gamma^{-1}$. The Hopf algebra $\mathcal{H}$ is called a quasi-triangular Yang-Baxter algebra if there is an element $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ called the universal R-matrix such that

$$\sigma \circ \Delta(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \tag{2.4}$$

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \tag{2.5a}$$

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.5b}$$

$$(\epsilon \otimes \mathrm{id})\mathcal{R} = 1 = (\mathrm{id} \otimes \epsilon)\mathcal{R} \tag{2.6a}$$

$$(\gamma \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{-1} = (\mathrm{id} \otimes \gamma^{-1})\mathcal{R} \tag{2.6b}$$

where $\mathcal{R}_{13}$ acts on the first and third spaces, $\mathcal{R}_{23}$ on the second and third and $\mathcal{R}_{12}$ on the first and second.

It follows from equations (2.4), (2.5a) and (2.5b) that the element $\mathcal{R}$ satisfies the Yang-Baxter equation given below

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{2.7}$$

The R-matrix and its properties, in particular the Yang-Baxter equation, are the key to the unique structure of q-deformed algebras and form the basis for many of their applications.

2.2 q-deformed algebras

The Lie algebras $su(n)$ can be defined in terms of a Chevalley basis $\{H_i, X_i^{\pm}\}$ where the i refers to the simple roots of the algebra. The generators have the following commutation relations

$$[H_i, H_j] = 0 \tag{2.8}$$

$$[H_j, X_i^{\pm}] = \pm a_{ij} X_i^{\pm} \tag{2.9}$$

$$[X_i^{+}, X_j^{-}] = \delta_{ij} H_i \tag{2.10}$$

where a_{ij} is an element of the Cartan matrix,

$$a_{ij} = \begin{cases} 2 & i - j = 0 \\ -1 & |i-j| = 1 \\ 0 & |i-j| \geq 2 \end{cases} \tag{2.11}$$

The q-deformed algebras $su(n)_q$ are defined by similar commutation relations to the $su(n)$ algebras

$$[H_i, H_j] = 0 \tag{2.12}$$

$$[H_j, X_i^{\pm}] = \pm a_{ij} X_i^{\pm} \tag{2.13}$$

$$[X_i^{+}, X_j^{-}] = \delta_{ij} [H_i]_q \tag{2.14}$$

$$(X_i^{\pm})^2 X_{i+1}^{\pm} - [2]_q X_i^{\pm} X_{i+1}^{\pm} X_i^{\pm} + X_{i+1}^{\pm} (X_i^{\pm})^2 = 0 \tag{2.15}$$

where the q-numbers or operators are defined as follows

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{2.16}$$

The subscript q will usually be dropped. In the limit as $q \to 1$, the q-numbers or operators $[x]_q$ are just x. Note that the differences in definition for $su(n)_q$ and $su(n)$ involve these q-operators, thus the q-deformed algebra $su(n)_q$ reduces to $su(n)$ in the $q = 1$ limit.

The notation varies somewhat in the literature. Many authors use $q' = q^{\pm 2}$ rather than q as used above. When comparing results in this thesis to those of other authors, we will note any differences in the definition of q. Some authors prefer to work with h where $q = e^h$. The limit $q \to 1$ becomes $h \to 0$ which is parallel to the quantum-classical transition.

The parameter q can be real or complex. Throughout this thesis, we will consider only the case where q is not a non-trivial root of unity. If q is a root of unity, the representations of $su(n)_q$ take on new characteristics and the representation theory changes considerably. Some representations become periodic or partially periodic (Arnaudon and Chakrabarti, 1991; Sun *et al*, 1990; Zhang, 1992). This leads to different solutions to the Yang-Baxter equation (Couture, 1991; Gómez *et al*, 1991; Hakobyan and Sedrakyan, 1993; Ruiz-Altaba, 1992). These representations do have some parallels with the standard case such as being described by q-bosons (Fu and Ge, 1992; Sun and Ge, 1991a, 1991b, 1991c, 1992). These representations have important applications particularly in conformal field theory (Bérkovich *et al*, 1993). The case $q = 0$ also has interesting properties but again will not be considered (Jimbo *et al*, 1991).

Equation (2.15) is the Serre relation for the q-deformed algebra. Rather than working in terms of this relation, further operators can be defined for each root analogous to those for the simple roots, but relations between these operators will be quadratic (Burroughs, 1990).

The q-deformed algebras are Hopf algebras having the following comultiplication, antipode and counit (Drinfel'd, 1986)

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{2.17a}$$
$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{H_i/2} + q^{-H_i/2} \otimes X_i^{\pm} \tag{2.17b}$$
$$\gamma(H_i) = -H_i \tag{2.18a}$$
$$\gamma(X_i^{\pm}) = -q^{\mp} X_i^{\pm} \tag{2.18b}$$
$$\epsilon(H_i) = 1 \tag{2.19a}$$
$$\epsilon(X_i)^{\pm} = 0 \tag{2.19b}$$

An alternative but equivalent definition for the antipode has $\gamma(X_i^{\pm}) = -q^{\rho} X_i^{\pm} q^{-\rho}$ (Reshetikhin, 1987) where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} H_\alpha$, Δ_+ being the set of positive roots for $su(n)$ (or $su(n)_q$). The Hopf algebra given above is the only possible one-parameter deformation of $su(n)$ within a scale of q.

The Hopf subalgebras generated by $\{H_i, X_i^+\}$ and $\{H_i, X_i^-\}$ are dual. Choosing a basis e_s for the first subalgebra and a corresponding dual basis e^s for the second, an R-matrix $\mathcal{R}$ can be defined for $su(n)_q$ as

$$\mathcal{R} = \sum_s e_s \otimes e^s \tag{2.20}$$

The R-matrix satisfies equations (2.4) to (2.6) (Drinfel'd, 1986). The q-deformed algebras $su(n)_q$ are thus quasi-triangular Yang-Baxter algebras.

In the following chapters we shall examine the similarities and differences between $su(n)$ and $su(n)_q$. The properties of R-matrices and their relation to the objects known in the $su(n)$ theory will be explored.

Chapter 3

Racah-Wigner algebra for $su(n)_q$

In this chapter we develop the Racah-Wigner algebra for $su(n)_q$ building on the work of Reshetikhin (1987), Nomura (1989) and others. Coupling coefficients can be defined for $su(n)_q$. They have more complicated symmetries than those found in $su(n)$ however. The R-matrices appear in symmetry relations. There are relationships between coupling and recoupling coefficients analogous to those of $su(n)$. Hence these can be calculated by recursion in a similar way to the $su(n)$ case. There is a relation between the coupling coefficients and R-matrices, the pentagonal equation, which allows the R-matrices to be calculated by recursion. These properties are discussed in Lienert and Butler (1992a) and the recursion method for R-matrices is introduced in Lienert and Butler (1992b). The specific examples in these two papers are given in the following two chapters.

3.1 Representation theory of $su(n)_q$

The Cartan subalgebra generated by $\{H_i\}$ is the same for $su(n)_q$ as for $su(n)$. The representation structure of $su(n)_q$ is therefore very similar to $su(n)$ (Rosso, 1988; Lusztig, 1988). Any representation of $su(n)_q$ has a corresponding $su(n)$ representation with the same dimension and weight spectrum.

Although the representations of $su(n)_q$ have the same dimensions as $su(n)$ in the sense of the number of weights of a representation, it is useful to define a q-deformed dimension as follows (Zhang *et al*, 1991; Alvarez-Gaume *et al*, 1990)

$$\begin{aligned} |\lambda|_q &= \sum_{\text{basis states}} q^{2\rho(\mu)} \\ &= \prod_{\alpha\in\Delta_+} \frac{[(\lambda+\rho,\alpha)]}{[(\rho,\alpha)]} \end{aligned} \tag{3.1}$$

where the sum is taken over all basis staes of the representation λ with due regard to weight multiplicity. In the $q \to 1$ limit, the first equation reduces to a trace over the weight space, while the second is the Weyl dimension formula. The q-subscript on $|\lambda|_q$ will usually be dropped.

3.2 Representation matrices

The representation space V^λ of a representation λ of $su(n)_q$ is spanned by the vectors $|\lambda i\rangle$ as for $su(n)$. The action of an element G of the algebra on the representation space is described by a representation matrix $D^\lambda(G)$

$$G\,|\lambda i\rangle = \sum_j |\lambda j\rangle\, D^\lambda_{ji}(G) \tag{3.2}$$

For $su(n)_q$ the elements of the representation matrices are non-commutative . Matrices with such non-commutative elements are an alternative starting point for the definition of quantum groups (Woronowicz, 1987a, 1987b; Manin, 1987; Matsuda *et al*, 1988; Groza *et al*, 1990).

The representation matrices are no longer strictly unitary. However, we define the complex conjugate matrix to $D^\lambda(G)$ (Nomura, 1990) so that

$$\sum_j D^\lambda_{ij}(G) D^\lambda_{kj}(G)^* = \sum_j D^\lambda_{ji}(G)^* D^\lambda_{jk}(G) = \delta_{ik} \tag{3.3}$$

The complex conjugate matrix $D^\lambda(G)^*$ is related to the matrix $D^{\lambda^*}(G)$ of the representation conjugate to λ.

3.3 Vector coupling coefficients

Two representations λ_1, λ_2 can be coupled together to give a third representation. The order of the coupling is important in $su(n)_q$. The coproduct of an algebra element is not in general invariant under an interchange of the spaces involved, see for example equation (2.18b).

The coupled representation can be reduced to a sum of irreducible representations, the coefficients involved are the vector coupling coefficients or Clebsch-Gordan coefficients

$$|\lambda_1 i_1\rangle\, |\lambda_2 i_2\rangle = \sum_{r\lambda i} |(\lambda_1\lambda_2) r\lambda i\rangle\ \ {}_q\langle r\lambda i|\lambda_1 i_1 \lambda_2 i_2\rangle \tag{3.4}$$

where r is a multiplicity label.

The coupling coefficients reduce the coupled representation matrix

$$\sum_{i_1 i_2 j_1 j_2} {}_q\langle r\lambda i|\lambda_1 j_1 \lambda_2 j_2\rangle D^{\lambda_1\lambda_2}_{j_1 i_1 j_2 i_2}(G)\; {}_q\langle \lambda_1 i_1 \lambda_2 i_2|r'\lambda' i'\rangle = \delta_{\lambda\lambda'}\delta_{rr'}D^{\lambda}_{ii'}(G) \tag{3.5}$$

where we define $D^{\lambda_1\lambda_2}$ so that

$$\Delta(G)\,|\lambda_1 i_1\rangle\,|\lambda_2 i_2\rangle = \sum_{j_1 j_2} |\lambda_1 j_1\rangle\,|\lambda_2 j_2\rangle\, D^{\lambda_1\lambda_2}_{j_1 i_1 j_2 i_2}(G) \tag{3.6}$$

The vector coupling coefficients form a unitary matrix

$$\sum_{i_1 i_2} {}_q\langle \lambda_1 i_1 \lambda_2 i_2|r\lambda i\rangle\; {}_q\langle r'\lambda' i'|\lambda_1 i_1 \lambda_2 i_2\rangle = \delta_{rr'}\delta_{\lambda\lambda'}\delta_{ii'} \tag{3.7}$$

$$\sum_{r\lambda i} {}_q\langle \lambda_1 i_1 \lambda_2 i_2|r\lambda i\rangle\; {}_q\langle r\lambda i|\lambda_1 i'_1 \lambda_2 i'_2\rangle = \delta_{i_1 i'_1}\delta_{i_2 i'_2} \tag{3.8}$$

where ${}_q\langle \lambda_1 i_1 \lambda_2 i_2|r\lambda i\rangle$ denotes ${}_q\langle r\lambda i|\lambda_1 i_1 \lambda_2 i_2\rangle^*$.

3.4 R-matrices

From the universal R-matrix, a matrix operator $R_q^{\lambda_1\lambda_2}$ acting to its right on the representation space $V^{\lambda_1}\otimes V^{\lambda_2}$ can be defined by

$$R_q^{\lambda_1\lambda_2} = P^{\lambda_1\lambda_2}\mathcal{R} \tag{3.9}$$

where $P^{\lambda_1\lambda_2}$ is the permutation operator mapping $V^{\lambda_1}\otimes V^{\lambda_2}$ to $V^{\lambda_2}\otimes V^{\lambda_1}$. It is this matrix we will generally call the R-matrix.

From equation (2.4), the R-matrices twist the product representation matrix

$$R_q^{\lambda_1\lambda_2} D^{\lambda_1\lambda_2}(G) = D^{\lambda_2\lambda_1}(G) R_q^{\lambda_1\lambda_2} \tag{3.10}$$

3.5 Properties of vector coupling coefficients

The properties of the $su(n)_q$ coupling coefficients are similar to those of $su(n)$ which are given by Butler (1975). They depend however on the deformation parameter q.

Reshetikhin (1987) gives the trivial vector coupling coefficient

$${}_q\langle \lambda\mu\, \lambda^* - \mu|00\rangle = \frac{q^{\rho(\mu)}}{|\lambda|^{1/2}}\phi_\mu \tag{3.11}$$

where ϕ is a phase and $\rho(\mu) = \frac{1}{2}\sum_{\alpha>0} H_\alpha(\mu)$, μ being a weight. The proof follows from the coproduct, equation (2.18b). The q terms appear due to the $q^{H/2}$ terms in the coproduct.

The symmetry of the vector coupling coefficients under interchange of the first two irreps is not trivial. From the coproduct, interchanging λ_1, λ_2 leads to a $q \to \frac{1}{q}$ change

$$ {}_q\langle \lambda_1 i_1 \lambda_2 i_2 | r \lambda i \rangle = \{(12)\lambda_1\lambda_2\lambda^*\}_{rs} \; {}_{\frac{1}{q}}\langle \lambda_2 i_2 \lambda_1 i_1 | s \lambda i \rangle \tag{3.12} $$

where $\{(12)\lambda_1\lambda_2\lambda^*\}_{rs}$ is the phase factor for this interchange. As for $su(n)$(Butler, 1975), we choose $\{(12)\lambda_1\lambda_2\lambda^*\}_{rs} = \{\lambda_1\lambda_2\lambda^*\}\,\delta_{rs}$.

In a similar manner to $su(n)$, the vector coupling coefficient and that obtained by replacing the irreps with their conjugates are related by (Reshetikhin, 1987)

$$ {}_q\langle \lambda_1 i_1 \lambda_2 i_2 | r \lambda i \rangle = \sum_{skl_1l_2} A(\lambda_1\lambda_2\lambda)_{rs} |\lambda|^{\frac{1}{2}} |\lambda_1|^{\frac{1}{2}} |\lambda_2|^{\frac{1}{2}} \; {}_q\langle 00 | \lambda i \lambda^* k \rangle \; {}_{\frac{1}{q}}\langle s \lambda^* k | \lambda_1^* l_1 \lambda_2^* l_2 \rangle $$
$$ {}_q\langle \lambda_1 i_1 \lambda_1^* l_1 | 00 \rangle \; {}_q\langle \lambda_2 i_2 \lambda_2^* l_2 | 00 \rangle \tag{3.13} $$

where for most algebras we can choose $A_{rs} = \delta_{rs}$ (Butler, 1975).

On interchanging λ_1 or λ_2 and λ_3, the vector coupling coefficients have the following symmetries

$$ \sum_{j_2} {}_q\langle \lambda_1 j_1 \lambda_2 j_2 | r \lambda_3 j_3 \rangle \; {}_q\langle 00 | \lambda_2 j_2 \lambda_2^* l \rangle = \frac{|\lambda_3|^{\frac{1}{2}}}{|\lambda_1|^{\frac{1}{2}} |\lambda_2|^{\frac{1}{2}}} \sum_s {}_q\langle s \lambda_1 j_1 | \lambda_3 j_3 \lambda_2^* l \rangle \; \{(13)\lambda_1\lambda_2\lambda_3^*\}_{rs} \tag{3.14} $$

$$ \sum_{j_1} {}_q\langle \lambda_1 j_1 \lambda_2 j_2 | r \lambda_3 j_3 \rangle \; {}_q\langle 00 | \lambda_1 j_1 \lambda_1^* l \rangle = \frac{|\lambda_3|^{\frac{1}{2}}}{|\lambda_1|^{\frac{1}{2}} |\lambda_2|^{\frac{1}{2}}} \sum_s {}_q\langle s \lambda_2 j_2 | \lambda_1^* l \lambda_3 j_3 \rangle \; \{(23)\lambda_1\lambda_2\lambda_3^*\}_{rs} \tag{3.15} $$

To prove the first statement, we consider the coupled representation matrix $D^{\lambda_1\lambda_2\lambda_2^*}(G)$. It can be reduced in two ways

$$ \sum_{\substack{i_1 i_2 \widehat{i_2} \\ j_1 j_2 \widehat{j_2}}} {}_q\langle s \lambda_1 k_1 | \lambda_3 i_3 \lambda_2^* \widehat{i_2} \rangle \; {}_q\langle r \lambda_3 i_3 | \lambda_1 i_1 \lambda_2 i_2 \rangle $$
$$ D^{\lambda_1\lambda_2\lambda_2^*}_{i_1 j_1 i_2 j_2 \widehat{i_2} \widehat{j_2}}(G) \; {}_q\langle \lambda_1 j_1 \lambda_2 j_2 | r \lambda_3 j_3 \rangle \; {}_q\langle \lambda_3 j_3 \lambda_2^* \widehat{j_2} | s' \lambda_1' l_1 \rangle = $$
$$ \sum_{\substack{i_3 \widehat{i_2} \\ j_3 \widehat{j_3}}} {}_q\langle s \lambda_1 k_1 | \lambda_3 i_3 \lambda_2^* \widehat{i_2} \rangle D^{\lambda_3\lambda_2^*}_{i_3 j_3 \widehat{i_2} \widehat{j_2}}(G) \; {}_q\langle \lambda_3 j_3 \lambda_2^* \widehat{j_2} | s' \lambda_1' l_1 \rangle = \delta_{ss'} \delta_{\lambda_1 \lambda_1'} D^{\lambda_1}_{k_1 l_1}(G) \tag{3.16} $$

or

$$ \sum_{\substack{m_1 m_2 \widehat{m_2} \\ n_1 n_2 \widehat{n_2}}} {}_q\langle \lambda_1 k_1 | \lambda_1 m_1 00 \rangle \; {}_q\langle 00 | \lambda_2 m_2 \lambda_2^* \widehat{m_2} \rangle $$
$$ D^{\lambda_1\lambda_2\lambda_2^*}_{m_1 n_1 m_2 n_2 \widehat{m_2} \widehat{n_2}}(G) \; {}_q\langle \lambda_2 n_2 \lambda_2^* \widehat{n_2} | 00 \rangle \; {}_q\langle \lambda_1 n_1 00 | \lambda_1 l_1 \rangle = $$
$$ \sum {}_q\langle \lambda_1 k_1 | \lambda_1 m_1 00 \rangle D^{\lambda_1 0}_{m_1 n_1 00}(G) \; {}_q\langle \lambda_1 n_1 00 | \lambda_1 l_1 \rangle = D^{\lambda_1}_{k_1 l_1}(G) \tag{3.17} $$

Equating these two schemes and using vector coupling coefficient orthogonality we have

$$\sum_{m_1 m_2 \widehat{m_2} i_3} {}_q\langle \lambda_1 i_1 \lambda_2 i_2 | r \lambda_3 i_3 \rangle \; {}_q\langle \lambda_3 i_3 \lambda_2^* \widehat{i_2} | s \lambda_1 \rangle$$
$${}_q\langle \lambda_1 k_1 | \lambda_1 m_1 0 0 \rangle \; {}_q\langle 0 0 | \lambda_2 m_2 \lambda_2^* \widehat{m_2} \rangle D^{\lambda_1 \lambda_2 \lambda_2^*}_{m_1 n_1 m_2 n_2 \widehat{m_2} \widehat{n_2}}(G) =$$
$$\sum_{j_1 j_2 \widehat{j_2} j_3} D^{\lambda_1 \lambda_2 \lambda_2^*}_{i_1 j_1 i_2 j_2 \widehat{i_2} \widehat{j_2}}(G) \; {}_q\langle \lambda_1 j_1 \lambda_2 j_2 | r \lambda_3 j_3 \rangle \; {}_q\langle \lambda_3 j_3 \lambda_2^* \widehat{j_2} | s \lambda_1 l_1 \rangle \; {}_q\langle 0 0 | \lambda_2 n_2 \lambda_2^* \widehat{n_2} \rangle \; {}_q\langle \lambda_1 l_1 | \lambda_1 n_1 0 \rangle \tag{3.18}$$

Schur's second lemma then states that the multiplier of $D^{\lambda_1 \lambda_2 \lambda_2^*}$ is a multiple of the identity and the result follows using equation (3.3). The second symmetry is established in a similar manner.

The R-matrices effect a $q \to 1/q$ transformation in the vector coupling coefficients (Reshetikhin, 1987),

$$\left(R_q^{\lambda_1 \lambda_2} \right)^{m_1 \, m_2}_{m_1' \, m_2'} \; {}_q\langle \lambda_1 m_1 \lambda_2 m_2 | r \lambda m \rangle = q^{c(\lambda_1) + c(\lambda_2) - c(\lambda)/2} \; {}_{\frac{1}{q}}\langle \lambda_1 m_1' \lambda_2 m_2' | r \lambda m \rangle \tag{3.19}$$

To prove this, we use equations (3.5) and (3.10). The product representation matrix $D^{\lambda_1 \lambda_2}(G)$ is written in terms of $D^{\lambda_2 \lambda_1}(G)$ and then substituting into equation (3.5)

$${}_q\langle s \lambda | \lambda_1 \lambda_2 \rangle \left(R_q^{\lambda_1 \lambda_2} \right)^{-1} D^{\lambda_2 \lambda_1}(G) R_q^{\lambda_1 \lambda_2} \; {}_q\langle \lambda_1 \lambda_2 | s' \lambda' \rangle = \delta_{\lambda \lambda'} \delta_{s s'} D^{\lambda}(G) \tag{3.20}$$

but $D^{\lambda}(G)$ can also be written as the decomposition of $D^{\lambda_2 \lambda_1}(G)$, so substituting and rearranging we have

$$D^{\lambda_2 \lambda_1}(G) R_q^{\lambda_1 \lambda_2} \; {}_q\langle \lambda_1 \lambda_2 | s' \lambda' \rangle \; {}_q\langle s' \lambda' | \lambda_2 \lambda_1 \rangle =$$
$$R_q^{\lambda_1 \lambda_2} \; {}_q\langle \lambda_1 \lambda_2 | s \lambda \rangle \; {}_q\langle s \lambda | \lambda_2 \lambda_1 \rangle D^{\lambda_2 \lambda_1}(G) \tag{3.21}$$

Schur's second lemma then implies

$$R^{\lambda_1 \lambda_2} \; {}_q\langle \lambda_1 \lambda_2 | s \lambda \rangle = {}_q\overline{R}_{\lambda} \; {}_q\langle \lambda_2 \lambda_1 | s \lambda \rangle \tag{3.22}$$

where ${}_q\overline{R}_{\lambda}$ is a multiple of the identity and $\overline{R}_{\lambda}|_{q=1} = \{\lambda_1 \lambda_2 \lambda^* s\}$. It follows from looking at R^{-1} that $\left({}_q\overline{R}_{\lambda} \right)^{-1} = {}_{\frac{1}{q}}\overline{R}_{\lambda}$ and hence

$${}_q\overline{R}_{\lambda} = \{\lambda_1 \lambda_2 \lambda^* s\} \, q^{\alpha} \tag{3.23}$$

To find α we expand the matrices in Taylor series about $q = 1$, with $R \to P + (q - 1) r + O((q-1)^2)$ where the classical r-matrix is given by (Drinfel'd, 1985)

$$r = \frac{1}{2} \sum_i H_i \otimes H_i + \sum_{\alpha \in \Delta^+} X_{\alpha}^+ \otimes X_{\alpha}^- + \sum_i X_{\alpha}^- \otimes X_{\alpha}^+ \tag{3.24}$$

and ${}_q\langle\lambda_1\lambda_2|s\lambda\rangle \to {}_{q=1}\langle\lambda_1\lambda_2|s\lambda\rangle + (q-1)\ {}_{q=1}\langle\lambda_1\lambda_2|s\lambda\rangle'$ and ${}_q\overline{R}_\lambda \to \{\lambda_1\lambda_2\lambda^* s\} + (q-1)\,\{\lambda_1\lambda_2\lambda^* s\}\,\alpha$.

Collecting terms of order $O(q-1)$ and eliminating two terms using symmetry we have

$$\alpha = \langle s\lambda|\lambda_1\lambda_2\rangle r\,\langle\lambda_1\lambda_2|s\lambda\rangle \tag{3.25}$$

Choosing the highest weight states since α is independent of weight state and acting with the r-matrix we obtain

$$\alpha = \frac{1}{2}(c(\lambda_1) + c(\lambda_2) - c(\lambda)) \tag{3.26}$$

where $c(\lambda_1)$ is the quadratic Casimir for the representation λ_1 of $su(n)$. This result together with equation (3.12) prove equation (3.19).

On using the orthogonality of the vector coupling coefficients, we have

$$\left(R_q^{\lambda_1\,\lambda_2}\right)^{m_1\,m_2}_{m_1'\,m_2'} = \sum_{r\lambda m} q^{c(\lambda_1)+c(\lambda_2)-c(\lambda)/2}\ {}_q\langle r\lambda m|\lambda_1 m_1\lambda_2 m_2\rangle\ {}_{\frac{1}{q}}\langle\lambda_1 m_1'\lambda_2 m_2'|r\lambda m\rangle \tag{3.27}$$

3.6 Properties of the R-matrices

The symmetries of the vector coupling coefficients imply the following symmetries for the R-matrices

$$\left(R_q^{\lambda_1\,\lambda_2}\right)^{m_1\,m_2}_{m_1'\,m_2'} = \left(R_q^{\lambda_2\,\lambda_1}\right)^{m_2'\,m_1'\ *}_{m_2\,m_1} \tag{3.28}$$

$$= \sum_{n_1'n_2'n_1n_2} \left(R_q^{\lambda_1^*\,\lambda_2^*}\right)^{n_1'\,n_2'}_{n_1\,n_2}\ {}_q\langle 00|\lambda_1 m_1\lambda_1^* n_1\rangle\ {}_{\frac{1}{q}}\langle 00|\lambda_1 m_1'\lambda_1^* n_1'\rangle|\lambda_1|$$
$${}_q\langle 00|\lambda_2 m_2\lambda_2^* n_2\rangle\ {}_{\frac{1}{q}}\langle 00|\lambda_2 m_2'\lambda_2^* n_2'\rangle|\lambda_2| \tag{3.29}$$

The R-matrices and vector coupling coefficients satisfy the pentagonal relation (Reshetikhin, 1987; Nomura, 1989; Hou *et al*, 1990a)

$$\sum_{m_1,m_2,m'} \left(R_q^{\lambda_1\,\lambda}\right)^{m_1\,m'}_{m_1'\,m''} \left(R_q^{\lambda_2\,\lambda}\right)^{m_2\,m}_{m_2'\,m'}\ {}_q\langle\lambda_1 m_1\lambda_2 m_2|r\lambda_3 m_3\rangle =$$
$$\sum_{m_3'} {}_q\langle\lambda_1 m_1'\lambda_2 m_2'|r\lambda_3 m_3'\rangle \left(R_q^{\lambda_3\,\lambda}\right)^{m_3\,m}_{m_3'\,m''} \tag{3.30}$$

This follows from the property of the universal R-matrix, equation (2.5a).

In matrix form the Yang-Baxter equation, (2.7), is

$$R_q^{\lambda_1\lambda_2} R_q^{\lambda_1\lambda_3} R_q^{\lambda_2\lambda_3} = R_q^{\lambda_2\lambda_3} R_q^{\lambda_1\lambda_3} R_q^{\lambda_1\lambda_2} \tag{3.31}$$

3.7 Recoupling coefficients and their properties

The recoupling coefficient arises when three representations λ_1, λ_2, λ_3 are coupled to give a fourth λ. The coupling can be done in two different manners: $(\lambda_1, \lambda_2) \to \lambda_{12}$, $(\lambda_{12}, \lambda_3) \to \lambda$ or $(\lambda_2, \lambda_3) \to \lambda_{23}$, $(\lambda_1, \lambda_{23}) \to \lambda$. The recoupling coefficient relates the two schemes. It is shown to satisfy

$$\sum_{r_{12}\lambda_{12}l_{12}r} {}_q\langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1 \lambda_2 l_2\rangle \; {}_q\langle r\lambda l|\lambda_{12}l_{12}\lambda_3 l_3\rangle$$
$$\times \; {}_q\langle \lambda_1, (\lambda_2\lambda_3)r_{23}\lambda_{23}; r'\lambda|(\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3; r\lambda\rangle$$
$$= \sum_{\lambda_{23}l_{23}} {}_q\langle r'\lambda l|\lambda_1 l_1 \lambda_{23} l_{23}\rangle \; {}_q\langle r_{23}\lambda_{23}l_{23}|\lambda_2 l_2 \lambda_3 l_3\rangle \quad (3.32)$$

The symmetries of the recoupling coefficients are easily derived from those of the vector coupling coefficients and are very similar to the $q = 1$ case although again they are dependent on q.

$${}_q\langle(\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3; r\lambda|\lambda_1, (\lambda_2\lambda_3)r_{23}\lambda_{23}; s\lambda\rangle$$
$$= \{(13)\lambda_1\lambda_2\lambda_{12}^*\}_{r_{12}r'_{12}} \{(13)\lambda_{12}\lambda_3\lambda^*\}_{rr'} \{(13)\lambda_1\lambda_{23}\lambda^*\}_{ss'} \{\lambda_2\lambda_3\lambda_{23}^* r_{23}\}$$
$$\times \; {}_{\frac{1}{q}}\langle(\lambda^*\lambda_3)r'\lambda_{12}^*, \lambda_2; r'_{12}\lambda_1^*|\lambda^*, (\lambda_3\lambda_2)r_{23}\lambda_{23}; s'\lambda_1^*\rangle$$
$$= \{(132)\lambda_1\lambda_2\lambda_{12}^*\}_{r_{12}r'_{12}} \{(13)\lambda_2\lambda_3\lambda_{23}^*\}_{r_{23}r'_{23}} \{(132)\lambda_1\lambda_{23}\lambda^*\}_{ss'} \{\lambda_{12}\lambda_3\lambda^* r\}$$
$$\times \; {}_q\langle(\lambda_{23}\lambda_3^*)r'_{23}\lambda_2, \lambda_{12}^*; r'_{12}\lambda_1^*|\lambda_{23}, (\lambda_3^*\lambda_{12}^*)r\lambda^*; s'\lambda_1^*\rangle$$
$$= \{(23)\lambda_1\lambda_2\lambda_{12}^*\}_{r_{12}r'_{12}} \{(23)\lambda_1\lambda_{23}\lambda^*\}_{ss'} \{(13)\lambda_{12}\lambda_3\lambda^*\}_{rr'} \{(13)\lambda_2\lambda_3\lambda_{23}^*\}_{r_{23}r'_{23}}$$
$$\times \; {}_q\langle(\lambda_1^*\lambda)s'\lambda_{23}, \lambda_3^*; r'_{23}\lambda_2|\lambda_1^*, (\lambda\lambda_3^*)r'\lambda_{12}; r'_{12}\lambda_2\rangle \quad (3.33)$$

Similar symmetries hold for other interchanges of the pairs (λ_1, λ_3^*), $(\lambda_{23}, \lambda_{12})$ and (λ^*, λ_2). Complex conjugation involves a change from q to $1/q$, as for the vector coupling coefficients

$${}_q\langle(\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3; r\lambda|\lambda_1, (\lambda_2\lambda_3)r_{23}\lambda_{23}; r'\lambda\rangle =$$
$${}_{\frac{1}{q}}\langle\lambda_1^*, (\lambda_2^*\lambda_3^*)r_{23}\lambda_{23}^*; r'\lambda^*|(\lambda_1^*\lambda_2^*)r_{12}\lambda_{12}^*, \lambda_3^*; r\lambda^*\rangle \quad (3.34)$$

The recoupling coefficients also satisfy the orthogonality relation

$$\sum_{\lambda_2 r_{12} r_{23}} \Big\{ {}_q\langle(\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3; r\lambda|\lambda_1, (\lambda_2\lambda_3)r_{23}\lambda_{23}s\lambda\rangle$$
$$\times \; {}_q\langle\lambda_1, (\lambda_2\lambda_3)r_{23}\lambda_{23}; s'\lambda'|(\lambda_1\lambda_2)r_{12}\lambda_{12}, \lambda_3; r'\lambda'\rangle \Big\} = \delta_{\lambda\lambda'}\delta_{rr'}\delta_{ss'} \quad (3.35)$$

Two expressions relating recoupling coefficients are the Racah backcoupling rule and the Biedenharn-Elliott sum rule. The Biedenharn-Elliott sum rule has no q-powers

$$\begin{aligned}
&\sum_t \Big\{ \{\lambda_1\lambda_{23}\lambda_{123}^* t\} \;{}_{\frac{1}{q}}\langle(\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3;r_{123}\lambda_{123}|\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23};t\lambda_{123}\rangle \\
&\qquad\qquad \times\; {}_q\langle(\lambda_{23}\lambda_1)t\lambda_{123},\lambda_4;r\lambda|\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14};s\lambda\rangle\Big\} \\
&= \sum_{r_{124}\lambda_{124}r'r'_{124}} \Big\{ \{\lambda_{12}\lambda_3\lambda_{123}^* r_{123}\}\{\lambda_{12}\lambda_4\lambda_{124}^* r_{124}\}\{\lambda_1\lambda_4\lambda_{14}^* r_{14}\}\{\lambda_{124}\lambda_3\lambda^* r'\}\{\lambda_{23}\lambda_{14}\lambda^* s\} \\
&\qquad \times\; {}_q\langle(\lambda_3\lambda_{12})r_{123}\lambda_{123},\lambda_4;r\lambda|\lambda_3,(\lambda_{12}\lambda_4)r_{124}\lambda_{124};r'\lambda\rangle \\
&\qquad \times\; {}_{\frac{1}{q}}\langle\lambda_4,(\lambda_1\lambda_2)r_{12}\lambda_{12};r_{124}\lambda_{124}|(\lambda_4\lambda_1)r_{14}\lambda_{14},\lambda_2;r'_{124}\lambda_{124}\rangle \\
&\qquad\qquad \times\; {}_{\frac{1}{q}}\langle(\lambda_{14}\lambda_2)r'_{124}\lambda_{124},\lambda_3;r'\lambda|\lambda_{14},(\lambda_2\lambda_3)r_{23}\lambda_{23};s\lambda\rangle\Big\} \qquad (3.36)
\end{aligned}$$

The Racah backcoupling rule for recoupling coefficients is

$$\begin{aligned}
&q^{\{-c(\lambda_1)-c(\lambda_2)-c(\mu_1)-c(\mu_2)\}}\;{}_q\langle(\lambda_1\mu_3)r_1\mu_2,\mu_1^*;r_3\lambda_3^*|\lambda_1,(\mu_3\mu_1^*)r_2\lambda_2;r_4\lambda_3^*\rangle \\
&= \sum_{rr'\nu} q^{\{-c(\nu)-c(\lambda_3)-c(\mu_3)\}}\{\mu_2\}\{\nu\}\{(13)\nu\mu_1^*\lambda_1^*\}_{rs}\{(132)\mu_2\lambda_2^*\nu\}_{r's'} \\
&\qquad \times \{(23)\lambda_2\mu_3\mu_1^*\}_{r_2s_2}\{\lambda_1\lambda_2\lambda_3r_4\}\;{}_q\langle(\lambda_2\nu)s'\mu_2,\mu_1^*;r_3\lambda_3^*|\lambda_2,(\nu\mu_1^*)s\lambda_1;r_4\lambda_3^*\rangle \\
&\qquad\qquad \times\; {}_q\langle(\lambda_1\mu_3)r_1\mu_2,\lambda_2^*;s'\nu|\lambda_1,(\mu_3\lambda_2^*)s_2\mu_1;s\nu\rangle \frac{|\nu||\lambda_2|^{1/2}}{\{|\mu_1||\lambda_1||\mu_2|\}^{1/2}} \qquad (3.37)
\end{aligned}$$

The Biedenharn-Elliott sum rule is proved by writing the recoupling coefficients in terms of vector coupling coefficients. Four coefficients are eliminated from the right-hand side and two from the left-hand side using orthogonality leaving identical coeffcients when the symmetry of equation (3.12) is used. On the left we have

$$\begin{aligned}
&\sum_t \Big\{ \{\lambda_1\lambda_{23}\lambda_{123}^* t\} \;{}_{\frac{1}{q}}\langle(\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3;r_{123}\lambda_{123}|\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23};t\lambda_{123}\rangle \\
&\qquad\qquad \times\; {}_q\langle(\lambda_{23}\lambda_1)t\lambda_{123},\lambda_4;r\lambda|\lambda_{23},(\lambda_1\lambda_4)r_{14}\lambda_{14};s\lambda\rangle\Big\} \\
&\quad = \sum \Big\{ \{\lambda_1\lambda_{23}\lambda_{123}^* t\}\;{}_{\frac{1}{q}}\langle r_{123}\lambda_{123}l_{123}|\lambda_{12}l_{12}\lambda_3l_3\rangle\;{}_{\frac{1}{q}}\langle r_{12}\lambda_{12}l_{12}|\lambda_1l_1\lambda_2l_2\rangle \\
&\quad {}_{\frac{1}{q}}\langle\lambda_2l_2\lambda_3l_3|r_{23}\lambda_{23}l_{23}\rangle\;{}_{\frac{1}{q}}\langle\lambda_1l_1\lambda_{23}l_{23}|t\lambda_{123}l_{123}\rangle\;{}_q\langle t\lambda_{123}l_{123}|\lambda_{23}l_{23}\lambda_1l_1\rangle \\
&\quad {}_q\langle r\lambda l|\lambda_{123}l_{123}\lambda_4l_4\rangle\;{}_q\langle\lambda_1l_1\lambda_4l_4|r_{14}\lambda_{14}l_{14}\rangle\;{}_q\langle\lambda_{23}l_{23}\lambda_{14}l_{14}|s\lambda l\rangle\Big\} \\
&\quad = \sum\;{}_q\langle\lambda_{23}l_{23}\lambda_1l_1|t\lambda_{123}l_{123}\rangle\;{}_q\langle t\lambda_{123}l_{123}|\lambda_{23}l_{23}\lambda_1l_1\rangle \\
&\quad {}_{\frac{1}{q}}\langle r_{123}\lambda_{123}l_{123}|\lambda_{12}l_{12}\lambda_3l_3\rangle\;{}_{\frac{1}{q}}\langle r_{12}\lambda_{12}l_{12}|\lambda_1l_1\lambda_2l_2\rangle\;{}_{\frac{1}{q}}\langle\lambda_2l_2\lambda_3l_3|r_{23}\lambda_{23}l_{23}\rangle \\
&\qquad\qquad {}_q\langle r\lambda l|\lambda_{123}l_{123}\lambda_4l_4\rangle\;{}_q\langle\lambda_1l_1\lambda_4l_4|r_{14}\lambda_{14}l_{14}\rangle\;{}_q\langle\lambda_{23}l_{23}\lambda_{14}l_{14}|s\lambda l\rangle\Big\} \qquad (3.38)
\end{aligned}$$

and on the right we have

$$\sum_{r_{124}\lambda_{124}r'r'_{124}} \Big\{ \{\lambda_{12}\lambda_3\lambda^*_{123}r_{123}\} \{\lambda_{12}\lambda_4\lambda^*_{124}r_{124}\} \{\lambda_1\lambda_4\lambda^*_{14}r_{14}\} \{\lambda_{124}\lambda_3\lambda^* r'\} \{\lambda_{23}\lambda_{14}\lambda^* s\}$$
$$\times\ {}_q\langle(\lambda_3\lambda_{12})r_{123}\lambda_{123}, \lambda_4; r\lambda|\lambda_3, (\lambda_{12}\lambda_4)r_{124}\lambda_{124}; r'\lambda\rangle$$
$$\times\ {}_{\frac{1}{q}}\langle\lambda_4, (\lambda_1\lambda_2)r_{12}\lambda_{12}; r_{124}\lambda_{124}|(\lambda_4\lambda_1)r_{14}\lambda_{14}, \lambda_2; r'_{124}\lambda_{124}\rangle$$
$$\times\ {}_{\frac{1}{q}}\langle(\lambda_{14}\lambda_2)r'_{124}\lambda_{124}, \lambda_3; r'\lambda|\lambda_{14}, (\lambda_2\lambda_3)r_{23}\lambda_{23}; s\lambda\rangle\Big\}$$
$$= \sum \{\lambda_{12}\lambda_3\lambda^*_{123}r_{123}\} \{\lambda_{12}\lambda_4\lambda^*_{124}r_{124}\} \{\lambda_1\lambda_4\lambda^*_{14}r_{14}\} \{\lambda_{124}\lambda_3\lambda^* r'\} \{\lambda_{23}\lambda_{14}\lambda^* s\}$$
$${}_q\langle r_{123}\lambda_{123}l_{123}|\lambda_3 l_3\lambda_{12}l_{12}\rangle\ {}_q\langle r\lambda l|\lambda_{123}l_{123}\lambda_4 l_4\rangle\ {}_q\langle\lambda_{12}l_{12}\lambda_4 l_4|r_{124}\lambda_{124}l_{124}\rangle$$
$${}_q\langle\lambda_3 l_3\lambda_{124}l_{124}|r'\lambda l\rangle\ {}_{\frac{1}{q}}\langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1\lambda_2 l_2\rangle\ {}_{\frac{1}{q}}\langle r_{124}\lambda_{124}l_{124}|\lambda_4 l_4\lambda_{12}l_{12}\rangle$$
$${}_{\frac{1}{q}}\langle\lambda_4 l_4\lambda_1 l_1|r_{14}\lambda_{14}l_{14}\rangle\ {}_{\frac{1}{q}}\langle\lambda_{14}l_{14}\lambda_2 l_2|r'_{124}\lambda_{124}l_{124}\rangle\ {}_{\frac{1}{q}}\langle r'_{124}\lambda_{124}l_{124}|\lambda_{14}l_{14}\lambda_2 l_2\rangle$$
$${}_{\frac{1}{q}}\langle r'\lambda l|\lambda_{124}l_{124}\lambda_3 l_3\rangle\ {}_{\frac{1}{q}}\langle\lambda_2 l_2\lambda_3 l_3|r_{23}\lambda_{23}l_{23}\rangle\ {}_{\frac{1}{q}}\langle\lambda_{14}l_{14}\lambda_{23}l_{23}|s\lambda l\rangle$$
$$= \sum {}_q\langle\lambda_{12}l_{12}\lambda_4 l_4|r_{124}\lambda_{124}l_{124}\rangle\ {}_q\langle r_{124}\lambda_{124}l_{124}|\lambda_{12}l_{12}\lambda_4 l_4\rangle$$
$${}_{\frac{1}{q}}\langle\lambda_{14}l_{14}\lambda_2 l_2|r'_{124}\lambda_{124}l_{124}\rangle\ {}_{\frac{1}{q}}\langle r'_{124}\lambda_{124}l_{124}|\lambda_{14}l_{14}\lambda_2 l_2\rangle$$
$${}_q\langle r_{123}\lambda_{123}l_{123}|\lambda_3 l_3\lambda_{12}l_{12}\rangle\ {}_q\langle r\lambda l|\lambda_{123}l_{123}\lambda_4 l_4\rangle$$
$${}_q\langle\lambda_3 l_3\lambda_{124}l_{124}|r'\lambda l\rangle\ {}_{\frac{1}{q}}\langle r_{12}\lambda_{12}l_{12}|\lambda_1 l_1\lambda_2 l_2\rangle\ {}_{\frac{1}{q}}\langle\lambda_4 l_4\lambda_1 l_1|r_{14}\lambda_{14}l_{14}\rangle$$
$${}_{\frac{1}{q}}\langle r'\lambda l|\lambda_{124}l_{124}\lambda_3 l_3\rangle\ {}_{\frac{1}{q}}\langle\lambda_2 l_2\lambda_3 l_3|r_{23}\lambda_{23}l_{23}\rangle\ {}_{\frac{1}{q}}\langle\lambda_{14}l_{14}\lambda_{23}l_{23}|s\lambda l\rangle \tag{3.39}$$

On using the orthogonality relations, equations (3.8) and (3.7), the two sides are equal.

The backcoupling rule follows from the identity

$${}_q\langle(\lambda_1\mu_3)r_1\mu_2, \mu^*_1; r_3\lambda^*_3|\lambda_1(\mu_3\mu^*_1)r_2\lambda_2; r_4\lambda^*_3\rangle$$
$$= \sum_{\nu r r'} {}_q\langle(\lambda_1\mu_3)r_1\mu_2, \mu^*_1; r_3\lambda^*_3|(\mu_2\lambda^*_2)r'\nu, \mu^*_1; r\lambda_1\rangle$$
$$\times\ {}_q\langle(\mu_2\lambda^*_2)r'\nu, \mu^*_1; r\lambda_1|\lambda_1(\mu_3\mu^*_1)r_2\lambda_2; r_4\lambda^*_3\rangle$$
$$= \sum_{\nu r r'} {}_q\langle\lambda_1 n_1\mu_2 m_3|r_1\mu_2 m_2\rangle\ {}_q\langle\mu_2 m_2\mu^*_1 l_1|r_3\lambda^*_3 k_3\rangle\ {}_q\langle r'\nu p|\mu_2 m_2\lambda^*_2 k_2\rangle$$
$$\times\ {}_q\langle r\lambda_1 n_1|\nu p\mu^*_1 l_1\rangle\ {}_q\langle\mu_2 m_2\lambda^*_2 k_2|r'\nu p\rangle\ {}_q\langle\nu p\mu^*_1 l_1|r\lambda_1 n_1\rangle$$
$$\times\ {}_q\langle r_4\lambda^*_3 k_3|\lambda_1 n_1\lambda_2 n_2\rangle\ {}_q\langle r_2\lambda_2 n_2|\mu_3 m_3\mu^*_1 l_1\rangle \tag{3.40}$$

where the right-hand side is expanded in terms of vector coupling coefficients. Using the symmetries proved for coupling coefficients to re-express four of the coupling coefficients

$$
\begin{aligned}
{}_q\langle r'\nu p|\mu_2 m_2\lambda_2^* k_2\rangle =& q^{\{c(\lambda_2)+c(\nu)-c(\mu_2)\}} \left(R_q^{\lambda_2\nu}\right)_{n_2' p}^{n_2'' p'} \; {}_q\langle\lambda_2 n_2''\nu p'|s'\mu_2 m_2\rangle \\
&\times \frac{|\nu|^{1/2}}{|\mu_2|^{1/2}} \; {}_q(\lambda_2^*)^{k_2 n_2'} \{\mu_2\} \{(132)\mu_2\lambda_2^*\nu\}_{r's'}
\end{aligned} \tag{3.41}
$$

from equations (3.13), (3.15), (3.14) and (3.19),

$$
{}_q\langle\nu p\mu_1^* l_1|r\lambda_1 n_1\rangle = {}_q\langle s\nu p|\lambda_1 n_1\mu_1 m_1'\rangle \frac{|\nu|^{1/2}}{|\lambda_1|^{1/2}} \; {}_q(\mu_1^*)_{l_1 m_1'} \{\nu\} \{(13)\nu\mu_1^*\lambda_1^*\}_{rs} \tag{3.42}
$$

from equations (3.13), (3.15), (3.14) and (3.12),

$$
\begin{aligned}
{}_q\langle r_4\lambda_3^* k_3|\lambda_1 n_1\lambda_2 n_2\rangle =& q^{\{c(\lambda_3)-c(\lambda_1)-c(\lambda_2)\}} \left(R_q^{\lambda_2\lambda_1}\right)_{n_2 n_1}^{n_2' n_1'} \\
&\times \; {}_q\langle r_4\lambda_3^* k_3|\lambda_2 n_2'\lambda_1 n_1'\rangle \{\lambda_1\lambda_2\lambda_3 r_4\}
\end{aligned} \tag{3.43}
$$

from equations (3.12) and (3.19)

$$
\begin{aligned}
{}_q\langle r_2\lambda_2 n_2|\mu_3 m_3\mu_1^* l_1\rangle =& q^{\{c(\mu_3)-c(\mu_1)-c(\lambda_2)\}} \left(R_q^{\mu_1\lambda_2}\right)_{m_1' n_2}^{m_1'' n_2'} \; {}_q\langle s_2\mu_1 m_1''|\mu_3 m_3\lambda_2^* k_2'\rangle \\
&\times \frac{|\lambda_2|^{1/2}}{|\mu_1|^{1/2}} \; {}_q(\mu_1)^{m_1' l_1} \; {}_q(\lambda_2^*)_{k_2' n_2'} \{(23)\lambda_2\mu_3\mu_1^*\}_{r_2 s_2}
\end{aligned} \tag{3.44}
$$

from equations (3.14), (3.12), (3.19) and (3.14). Substituting back into equation (3.44) and using the pentagonal relation, (3.30), the R-matrices cancel giving

$$
\begin{aligned}
&{}_q\langle(\lambda_1\mu_3)r_1\mu_2, \mu_1^*; r_3\lambda_3^*|\lambda_1(\mu_3\mu_1^*)r_2\lambda_2; r_4\lambda_3^*\rangle \\
&= \sum_{\nu r r'} {}_q\langle\lambda_1 n_1\mu_3 m_3|r_1\mu_2 m_2\rangle \; {}_q\langle\mu_2 m_2\mu_1^* l_1|r_3\lambda_3^* k_3\rangle \; {}_q\langle\mu_2 m_2\lambda_2^* k_2|r'\nu p\rangle \\
&\quad\times \; {}_q\langle s\nu p'|\lambda_1 n_1'\mu_1 m_1''\rangle \; {}_q\langle\lambda_2 n_2''\nu p'|s'\mu_2 m_2\rangle \; {}_q\langle r\lambda_1 n_1|\nu p\mu_1^* l_1\rangle \\
&\quad\times \; {}_q\langle r_4\lambda_3^* k_3|\lambda_2 n_2'\lambda_1 n_1'\rangle \; {}_q\langle s_2\mu_1 m_1''|\mu_3 m_3\lambda_2^* k_2'\rangle \frac{|\nu||\lambda_2|^{\frac{1}{2}}}{\{|\mu_1||\lambda_1||\mu_2|\}^{\frac{1}{2}}} \\
&\quad\times \{\lambda_1\lambda_2\lambda_3^* r_4\} \{(132)\mu_2\lambda_2^*\nu\}_{r's'} \{(13)\nu\mu_1^*\lambda_1^*\}_{rs} \{(23)\lambda_2\mu_3\mu_1^*\}_{r_2 s_2} \{m_2\}\{\nu\} \\
&\quad\times q^{\{c(\nu)+c(\lambda_3)+c(\mu_3)-c(\lambda_1)-c(\lambda_2)-c(\mu_1)-c(\mu_2)\}} \\
&= \sum_{\nu r r'} q^{\{c(\nu)+c(\lambda_3)+c(\mu_3)-c(\lambda_1)-c(\lambda_2)-c(\mu_1)-c(\mu_2)\}} \{\mu_2\} \{\nu\} \{(13)\nu\mu_1^*\lambda_1^*\}_{rs} \{(132)\mu_2\lambda_2^*\nu\}_{r's'} \\
&\quad\times \{(23)\lambda_2\mu_3\mu_1^*\}_{r_2 s_2} \{\lambda_1\lambda_2\lambda_3 r_4\} \; {}_q\langle(\lambda_2\nu)s'\mu_2, \mu_1^*; r_3\lambda_3^*|\lambda_2, (\nu\mu_1^*)s\lambda_1; r_4\lambda_3^*\rangle \\
&\qquad\qquad\times \; {}_q\langle(\lambda_1\mu_3)r_1\mu_2, \lambda_2^*; s'\nu|\lambda_1, (\mu_3\lambda_2^*)s_2\mu_1; s\nu\rangle \frac{|\nu||\lambda_2|^{\frac{1}{2}}}{\{|\mu_1||\lambda_1||\mu_2|\}^{\frac{1}{2}}}
\end{aligned} \tag{3.45}
$$

It is useful to work with the q-$6j$ symbols instead of the recoupling coefficients as the symmetries are easier to apply. A q-$3jm$ symbol could be defined but does not have useful symmetries unless a q factor is included. The q-$6j$ symbol is defined in terms of the recoupling coefficient by

$$ {}_q\begin{Bmatrix} \lambda_1 & \lambda_{23} & \lambda^* \\ \lambda_3^* & \lambda_{12} & \lambda_2 \end{Bmatrix}_{r s_{23} s_{12} r'} = \{|\lambda_{12}||\lambda_{23}|\}^{-1/2} \{\lambda_2\} \{\lambda_{12}\lambda_3\lambda^* r\} \{(23)\lambda_1\lambda_2\lambda_{12}^*\}_{r_{12}s_{12}} $$
$$ \times \{(123)\lambda_2\lambda_3\lambda_{23}^*\}_{r_{23}s_{23}} \; {}_q\langle(\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3; r\lambda|\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23}; r'\lambda\rangle \quad (3.46) $$

The symmetries of the q-$6j$ symbol are easily derived from those of the recoupling coefficients and are very similar to the $q = 1$ case. Interchanging two columns of the q-$6j$ symbol leads to a phase factor, while interchanging two of each row does not

$$ {}_q\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \{\lambda_1\mu_2\mu_3^* r\} \{\mu_1^*\lambda_2\mu_3 r_2\} \{\mu_1\mu_2^*\lambda_3 r_3\} \{\lambda_1\lambda_2\lambda_3 r_4\} $$
$$ \times \{\mu_1\}\{\mu_2\}\{\mu_3\} \, {}_q\begin{Bmatrix} \lambda_2 & \lambda_1 & \lambda_3 \\ \mu_2^* & \mu_1^* & \mu_3^* \end{Bmatrix}_{r_2r_1r_3r_4} \quad (3.47) $$
$$ = {}_q\begin{Bmatrix} \lambda_1^* & \mu_2 & \mu_3^* \\ \mu_1^* & \lambda_2 & \lambda_3^* \end{Bmatrix}_{r_4r_3r_2r_1} \quad (3.48) $$

Complex conjugation involves a change from q to $1/q$, as for the vector coupling coefficients

$$ {}_{\frac{1}{q}}\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r_1r_2r_3r_4} = {}_q\begin{Bmatrix} \lambda_1^* & \lambda_2^* & \lambda_3^* \\ \mu_1^* & \mu_2^* & \mu_3^* \end{Bmatrix}_{r_1r_2r_3r_4} \quad (3.49) $$

The q-$6j$ symbols also satisfy the orthogonality relation

$$ \sum_{\mu_3 r_1 r_2} |\lambda_3||\mu_3| \, {}_q\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} {}_q\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3' \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}^*_{r_1r_2r_3'r_4'} = $$
$$ \delta_{\lambda_3\lambda_3'}\delta_{r_3r_3'}\delta_{r_4r_4'} \quad (3.50) $$

Equation (3.32) can be rearranged into a form able to be used in recursively calculating the vector coupling coefficients,

$$ \sum {}_q\langle\lambda_3^* l_3|\lambda_1 l_1 \lambda_2 l_2\rangle \, {}_q\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} $$
$$ = \sum (\mu_1)^{m_1m_1'}(\mu_2)^{m_2m_2'}(\mu_3)^{m_3m_3'}|\mu_3| \{\lambda_1\mu_2\mu_3\} \; {}_{\frac{1}{q}}\langle\lambda_3^* l_3|\mu_1^* m_1' \mu_2 m_2\rangle $$
$$ \times \; {}_{\frac{1}{q}}\langle\mu_3 m_3|\mu_1 m_1 \lambda_2 l_2\rangle \; {}_{\frac{1}{q}}\langle\mu_3^* m_3'|\lambda_1 l_1 \mu_2^* m_2\rangle \quad (3.51) $$

3.8 Recursion techniques

The coupling and recoupling coefficients for $su(n)_q$ can be calculated in a similar manner to that developed for $su(n)$ and other compact Lie groups by Butler and Wybourne (1976). The trivial vector coupling coefficient, i.e. that containing the scalar irrep, is known, equation (3.11). The trivial recoupling coefficient can be obtained directly from this. A 'primitive' irrep ϵ is chosen, usually the lowest dimension faithful irrep. The power of any other irrep λ is then defined as the lowest n such that $\lambda \in (\epsilon \oplus \epsilon *)^{\otimes n}$.

Primitive coupling coefficients, i.e. those containing the primitive irrep, can be obtained from their orthogonality and symmetry properties, equations (3.8) and (3.7). Similarily, primitive $6j$-symbols can be obtained from orthogonality, equation (3.54) and the Racah-backcoupling equation, equation (3.37).

The general $6j$-symbols are obtained from the Biedenharn-Elliot sum rule by choosing one of the irreps on the right-hand-side to be the primitive irrep. The remaining irreps on the right-hand side can then be chosen to be of lower power than those on the left.

The R-matrices can also be calculated by a recursion technique. The method used by Nomura (1989) and Ma (1990a, 1990b) using equation (3.27) involves a sum over a product of matrices. The coupling coefficients must be known, but often it is the R-matrices only which are of interest not the coupling coefficients. Using recursion, only the primitive coupling coefficients need to be calculated. The trivial and primitive R-matrices are then obtained directly from the trivial and primitive coupling coefficients with equation (3.27). Choosing λ_1 to be the primitive irrep ϵ in the pentagonal equation, equation (3.30), gives

$$R_q^{\epsilon\lambda} R_q^{\lambda_2\lambda} \; {}_q\langle \epsilon\lambda_2 | \lambda_3 \rangle = {}_q\langle \epsilon\lambda_2 | \lambda_3 \rangle R_q^{\lambda_3\lambda} \tag{3.52}$$

The primitive coupling coefficients and primitive R-matrices are known so that equation (3.56) gives $R^{\lambda_2\lambda}$ in terms of $R^{\lambda_3\lambda}$. The representation λ_2 can always be chosen to have power $p(\lambda_2) = p(\lambda_3) - 1$. The recursion can be continued with eventually the trivial R-matrix $R^{\{0\}\lambda}$ being substituted for. We give examples of these recursive techniques in the next two chapters.

Chapter 4

The q-deformed algebra $su(2)_q$

In the next two chapters, we consider explicitly two q-deformed algebras. The recursive methods developed in the previous chapter are used to obtain expressions for the $6j$-symbols, coupling coefficients and R-matrices of $su(2)_q$ and classes of the coupling coefficients and R-matrices of $su(3)_q$.

In the case of $su(2)_q$, expressions are already known for all of the coupling coefficients, $6j$-symbols and R-matrices. The coupling coefficients and $6j$-symbols were first obtained in Kirillov and Reshetikhin (1988). The authors followed the same method as that first used in the equivalent $su(2)$ calculations (Wigner, 1931; Racah, 1942; Edmonds, 1957). The coupling coefficients are obtained by looking at the action of the operators on the basis. The $6j$-symbols are found directly from their definition in terms of coupling coefficients, the summations in the product of coupling coefficients being successively reduced in a lengthy process. Hou *et al* (1990a, 1990b) followed a similar method to obtain coupling coefficients and $6j$-symbols. Other authors have used the properties of hypergeometric functions to obtain $6j$-symbols and coupling coefficients (Groza *et al*, 1990; Kachurik and Klimyk, 1990). Various other methods have been used generalizing those used in the $su(2)$ calculation (Avancini and Menezes, 1993; Koelink and Koornwinder, 1989; Ruegg, 1990). Recurrence relations for both $6j$-symbols and coupling coefficients have been found but from consideration of the known form of the coefficients (Kachurik and Klimyk, 1991). We illustrate the calculation of $su(2)_q$ coupling coefficients and $6j$-symbols by a new method. This method, unlike most of those used above, can be generalized to $su(n)_q$.

Although the operator form of the R-matrix is known in many cases, very few matrix elements have been calculated. Nomura (1989) presents an expression for the R-matrix of $su(2)_q$ obtained by looking at a limit of the known expression for the $6j$-symbol. It is different in its powers of q to that obtained by Kirillov and Reshetikhin (1988). We illustrate a new recursive method of calculating R-matrices in this chapter.

4.1 Identities for q-numbers

The calculation of the various coefficients requires the manipulation of q-numbers, defined in equation (2.16). As in the study of q-series, a q-factorial and q-binomial coefficient can be defined (Andrews, 1976)

$$[n]! = [1][2]\dots[n], \quad \begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[n-r]![r]!} \tag{4.1}$$

The following identities containing q-numbers are useful

$$[n] = q^{n-1} + q^{n-3} + \dots + q^{-n+1} \tag{4.2}$$

$$q^{\pm n}[n'] + q^{\mp n'}[n] = [n+n'] \tag{4.3}$$

$$[n][n'+n''] - [n+n''][n'] = [n-n'][n''] \tag{4.4}$$

$$q^{\pm(n+1-k)} \begin{bmatrix} n \\ k-1 \end{bmatrix} + q^{\mp k} \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ k \end{bmatrix} \tag{4.5}$$

$$\sum_{k=1}^{n} [k] q^{\pm 3k} = q^{\pm(2n+1)} \frac{[n][n+1]}{[2]} \tag{4.6}$$

$$\sum_{\rho} \begin{bmatrix} n \\ \rho \end{bmatrix} \begin{bmatrix} m \\ r-\rho \end{bmatrix} q^{\pm n(r-\rho)\mp\rho m} = \begin{bmatrix} n+m \\ r \end{bmatrix} \tag{4.7}$$

$$\sum_{\tau} (-)^{\tau} \frac{[p-\tau]!}{[\tau]![n-\tau]![r-\tau]!} q^{\pm\tau(p-r-n+1)} = \frac{[p-n]![p-r]!}{[n]![r]![p-n-r]!} q^{\mp nr} \tag{4.8}$$

Equations (4.2) and (4.3) follow directly from the definition of a q-number, equation (2.16). The following two equations, (4.4) and (4.5), follow directly from those preceding. Equations (4.6), (4.7) and (4.8) can be established by induction.

To illustrate, we prove equation (4.7) using induction on m as follows. For $m = 1$, equation (4.7) reduces to equation (4.5). Suppose (4.7) is true for $m = k$, i.e.

$$\sum_\rho \begin{bmatrix} n \\ \rho \end{bmatrix} \begin{bmatrix} k \\ r-\rho \end{bmatrix} q^{n(r-\rho)-\rho k} = \begin{bmatrix} n+k \\ r \end{bmatrix} \tag{4.9}$$

then when $m = k$ we have

$$\begin{aligned}
&\sum_\rho \begin{bmatrix} n \\ \rho \end{bmatrix} \begin{bmatrix} k+1 \\ r-\rho \end{bmatrix} q^{n(r-\rho)-\rho(k+1)} \\
&= \sum_\rho \begin{bmatrix} n \\ \rho \end{bmatrix} \left\{ q^{k-r+\rho+1} \begin{bmatrix} k \\ r-\rho-1 \end{bmatrix} + q^{-r+\rho} \begin{bmatrix} k \\ r-\rho \end{bmatrix} \right\} q^{n(r-\rho)-\rho(k+1)} \\
&= \left\{ \sum_\rho \begin{bmatrix} n \\ \rho \end{bmatrix} \begin{bmatrix} k \\ r-\rho-1 \end{bmatrix} q^{n(r-1-\rho)-\rho k} \right\} q^{n+k+1-r} + \left\{ \sum_\rho \begin{bmatrix} n \\ \rho \end{bmatrix} \begin{bmatrix} k \\ r-\rho \end{bmatrix} q^{n(r-\rho)-\rho k} \right\} q^{-r} \\
&= \begin{bmatrix} n+k \\ r-1 \end{bmatrix} q^{n+k+1-r} + \begin{bmatrix} n+k \\ r \end{bmatrix} q^{-r} \\
&= \begin{bmatrix} n+k+1 \\ r \end{bmatrix}
\end{aligned} \tag{4.10}$$

where we have used equation (4.5). Hence the result is true by induction.

4.2 Trivial coefficients

The q-dimension of an irrep j of $su(2)_q$ is given by $|j| = q^{2j} + q^{2j-2} + \ldots q^{-2j} = [2j+1]$ from the definition of the q-dimension, equation (3.1) and the relation for the q-number, equation (4.2). Note that the notation conflicts with that sometimes used in quantum angular momentum theory; $[j]$ is used here for the q-number j not the dimension of the representation j.

The trivial $su(2)_q$ vector coupling coefficient from equation (3.11) is

$$_q\langle jmjm'|00\rangle = (-)^{j-m}\delta_{m-m'}q^m[2j+1]^{-\frac{1}{2}} \tag{4.11}$$

Substituting into the definition of the recoupling coefficient and $6j$-symbol, the trivial $su(2)_q$ $6j$ symbol is

$$_q\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_2 & j_1 & 0 \end{Bmatrix} = \frac{(-)^{j_1+j_2+j_3}}{\{[2j_1+1][2j_2+1]\}^{\frac{1}{2}}} \tag{4.12}$$

In a similar manner the trivial R-matrix is

$$(R^{0j})^{0m}_{0m} = 1 \tag{4.13}$$

The eigenvalue of the quadratic Casimir of an irrep j is $c(j) = j(j+1)$.

4.3 Primitive $6j$-symbols

The primitive irrep for $su(2)_q$ is that for which $j = \frac{1}{2}$ so that the primitive $6j$-symbols are of one of the two forms

$$\left\{ \begin{matrix} a & \frac{1}{2} & a+\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{matrix} \right\}_q, \left\{ \begin{matrix} a & \frac{1}{2} & a-\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{matrix} \right\}_q \tag{4.14}$$

The calculation for $6j$-symbols of $su(2)_q$ is carried out in exactly the same manner as for $su(2)$ (Butler 1976). The orthogonality properties (3.54) give three equations in the primitive $6j$-symbols. On combining the equations using symmetries, equations (3.52) and (3.52), we obtain the relation

$$[2a+2][2b+1]\left\{ \begin{matrix} a & \frac{1}{2} & a+\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{matrix} \right\}_q^2 = [2a][2b-1]\left\{ \begin{matrix} a & \frac{1}{2} & a-\frac{1}{2} \\ b-\frac{1}{2} & c & b-1 \end{matrix} \right\}_q^2 \tag{4.15}$$

and on iteration

$$\left\{ \begin{matrix} a & \frac{1}{2} & a+\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{matrix} \right\}_q^2 = \frac{[2a]![2b-1]![2a+2-x]![2b+1-x]!}{[2a-x]![2b-1-x]![2a+2]![2b+1]!} \times \left\{ \begin{matrix} a-\frac{x}{2} & \frac{1}{2} & a-\frac{x}{2}+\frac{1}{2} \\ b-\frac{x}{2}-\frac{1}{2} & c & b-\frac{x}{2} \end{matrix} \right\}_q^2 \tag{4.16}$$

The boundary condition occurs when x has its maximum value satisfying the triangle conditions, $x = a+b-c$. Then the orthogonality condition, (3.54), gives for the boundary $6j$-symbol

$$\left\{ \begin{matrix} \frac{1}{2}(a-b+c) & \frac{1}{2} & \frac{1}{2}(a-b+c+1) \\ \frac{1}{2}(-a+b+c-1) & c & \frac{1}{2}(-a+b+c) \end{matrix} \right\}_q^2 = \frac{1}{[a-b+c+2][-a+b+c+1]} \tag{4.17}$$

Substituting back, cancelling terms and taking the square root

$$\left\{ \begin{matrix} a & \frac{1}{2} & a+\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{matrix} \right\}_q = \alpha_{abc}\left\{ \frac{[a-b+c+1][-a+b+c]}{[2a+1][2a+2][2b][2b+1]} \right\}^{\frac{1}{2}} \tag{4.18}$$

where α_{abc} is a phase to be determined. Using (4.18) and the orthogonality condition (3.54) together with a q-number identity, equation (4.4), gives the second primitive $6j$-symbol

$$\left\{ \begin{matrix} a & \frac{1}{2} & a-\frac{1}{2} \\ b-\frac{1}{2} & c & b \end{matrix} \right\}_q = \beta_{abc}\left\{ \frac{[a+b-c][a+b+c+1]}{[2a][2a+1][2b][2b+1]} \right\}^{\frac{1}{2}} \tag{4.19}$$

The Racah backcoupling rule (3.37) together with the phase of the trivial $6j$-symbol and the symmetries of the $6j$-symbols enable the phases α_{abc} and β_{abc} to be found

$$\alpha_{abc} = \beta_{abc} = (-)^{a+b+c} \tag{4.20}$$

4.4 Calculation of general 6j-symbols

The general 6j-symbols are found from the primitive 6j-symbols by using the Biedenharn–Elliott sum rule, equation (3.36). Substituting the primitive 6j-symbol, (4.18), into the right-hand side of equation (3.36) and simplifying using symmetries and the substitution

$$\left|\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right|_q = \frac{[p+r-c]![a-r+s]![-p+b+s]!}{[a+r+s+1]![b+p+s+1]!} \times \{\Delta(abc)\Delta(aps)\Delta(bps)\Delta(prc)\}^{-1} \left\{\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right\}_q \tag{4.21}$$

where

$$\Delta(abc) = \left\{\frac{[a+b-c]![a-b+c]![-a+b+c]!}{[a+b+c+1]!}\right\}^{\frac{1}{2}} \tag{4.22}$$

gives the recursion relation

$$[2s+1]\left|\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right|_q = \left|\begin{matrix} a & b & c \\ p-\frac{1}{2} & r-\frac{1}{2} & s+\frac{1}{2} \end{matrix}\right|_q - \left|\begin{matrix} a & b & c \\ p-\frac{1}{2} & r-\frac{1}{2} & s-\frac{1}{2} \end{matrix}\right|_q \tag{4.23}$$

On iterating, we obtain

$$\left|\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right|_q = \sum_{x=0}^{X}(-)^x \frac{[X]![2s-x]!}{[x]![X-x]![2s+1+X-x]!} \times [2s+1+X-2x]\left|\begin{matrix} a & b & c \\ p-\frac{X}{2} & r-\frac{X}{2} & s+\frac{X}{2}-x \end{matrix}\right|_q \tag{4.24}$$

This solution is verified by substituting into equation (4.23) and combining the two terms on the right-hand side using the q-number identity equation (4.5).

The 6j-symbol on the right-hand side of (4.24) has a stretched form when X has the maximum value $X = p + r - c$. The 6j-symbol related to this stretched form by symmetry (3.52) can be found from equation (4.24) with $X = 2A$

$$\left|\begin{matrix} a & s' & B \\ A & A+B & b \end{matrix}\right|_q = \sum_{y=0}^{2A}(-)^y \frac{[2A]![2b-y]!}{[y]![2A-y]![2b+1+2A-y]!} \times [2b+1+2A-2y]\left|\begin{matrix} a & s' & B \\ 0 & B & b+A-y \end{matrix}\right|_q \tag{4.25}$$

The only non-zero term on the right is that for which $y = b + A - s'$. Substituting the value of the trivial 6j, (4.12), and the definition (4.21) into (4.25) we obtain for the stretched $6j$-symbol

$$\left|\begin{matrix} a & b & A+B \\ A & B & s' \end{matrix}\right|_q = \frac{(-)^{a+b+A+B}[b-A+s']!}{[A-b+s']![b+A-s']![-a+s'+B]!} \times \frac{[a+b+A+B+1]!}{[a-s'+b]![a+b-A-B]!} \tag{4.26}$$

Substituting into (4.24), using symmetries and a further substitution of $y = p+r+c+x$ yields an explicit form for the q-$6j$-symbols

$$\left\{\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right\}_q = \sum_y (-)^y [a+b+c+1+2(p+r+s-y)] \times \Delta(abc)\Delta(ars)\Delta(pbs)\Delta(prc) \times \prod_{\text{pairs}} \left\{ \frac{[a+r+s+1]![a+r+s+2p-y]!}{[-a+r+s]![b+c+r+s+2p+1-y]!} \times \frac{1}{[y-a-r-s]![b+c+r+s-y]!} \right\} \tag{4.27}$$

where the product is over cyclic permutations of the pairs (a,p), (b,r) and (c,s).

The form of the $6j$-symbol obtained here is different to that of Kirillov and Reshetikhin (1988) and Hou *et al* (1990a, 1990b). Both forms share the property of being related to $q=1$ expressions simply by substituting q-factorials for factorials. This result is not unexpected. The $su(2)$ $6j$-symbols are real, so for real q the $su(2)_q$ $6j$-symbols will be real. Together with the self-adjoint property of the $su(2)_q$ representations, this implies from equation (3.53)

$$\left\{\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right\}_q = \left\{\begin{matrix} a & b & c \\ p & r & s \end{matrix}\right\}_{\frac{1}{q}} \tag{4.28}$$

Hence the $su(2)_q$ $6j$-symbols must contain only terms symmetric in q and q^{-1} as is the case for q-numbers.

4.5 Primitive vector coupling coefficients

The two orthogonality relations (3.8) and (3.7) with $\lambda_2 = \frac{1}{2}$ give two two-term equations from which one coupling coefficient with $m_2 = -\frac{1}{2}$ can be eliminated using the symmetry (3.14). On rearranging, changing notation and simplifying we have

$$|\,{}_q\langle jm\,\tfrac{1}{2}\tfrac{1}{2}|j+\tfrac{1}{2}\,m+\tfrac{1}{2}\rangle|^2 = \frac{q^{2j}}{[2j+1]} + \frac{q^{-1}[2j]}{[2j+1]}|\,{}_q\langle j-\tfrac{1}{2}\,m-\tfrac{1}{2}\,\tfrac{1}{2}\tfrac{1}{2}|jm\rangle|^2 \tag{4.29}$$

On iterating k times we have

$$|\,{}_q\langle jm\,\tfrac{1}{2}\tfrac{1}{2}|j+\tfrac{1}{2}\,m+\tfrac{1}{2}\rangle|^2 = \frac{q^{2j+1-k}[k]}{[2j+1]} + \frac{q^{-k}[2j+1-k]}{[2j+1]}|\,{}_q\langle j-\tfrac{k}{2}\,m-\tfrac{k}{2}\,\tfrac{1}{2}\tfrac{1}{2}|j-\tfrac{k-1}{2}\,m-\tfrac{k-1}{2}\rangle|^2 \quad (4.30)$$

For $k = j+m+1$ the coefficient on the right-hand side ${}_q\langle\frac{j-m-1}{2}\,\frac{-j+m-1}{2}\,\frac{1}{2}\frac{1}{2}|\frac{j-m}{2}\,\frac{-j+m}{2}\rangle$ is zero since $|\frac{j-m-1}{2}| < |\frac{-j+m-1}{2}|$. On substituting into equation (4.30) we have

$$|\,{}_q\langle jm\,\tfrac{1}{2}\tfrac{1}{2}|j+\tfrac{1}{2}\,m+\tfrac{1}{2}\rangle|^2 = \frac{q^{j-m}[j+m+1]}{[2j+1]} \quad (4.31)$$

Finally choosing a phase to correspond with that used in $su(2)$

$${}_q\langle jm\,\tfrac{1}{2}\tfrac{1}{2}|j+\tfrac{1}{2}\,m+\tfrac{1}{2}\rangle = q^{(j-m)/2}\left\{\frac{[j+m+1]}{[2j+1]}\right\}^{\frac{1}{2}} \quad (4.32)$$

All other primitive vector coupling coefficients are related to the one given above by symmetry.

4.6 Calculation of general vector coupling coefficients

Letting $\mu_3 = \frac{1}{2}$ in the recoupling equation, (3.55), gives two terms on the right-hand side. After substituting for the primitive $6j$-symbol and primitive vector coupling coefficients from equations (4.19) and (4.32), simplifying and changing notation, $\lambda_1 \to j_1$ etc., the following recursion relation is obtained

$$\begin{aligned}{}_q\langle j_1m_1j_2m_2|j_3m_3\rangle &= \{[j_1+j_2-j_3][j_1+j_2+j_3+1]\}^{\frac{1}{2}} \\ &\times\Big\{\{[j_1+m_1][j_2-m_2]\}^{\frac{1}{2}}\,q^{j_1+j_2-m_1+m_2+1/2}\,{}_q\langle j_1-\tfrac{1}{2}\,m_1-\tfrac{1}{2}\,j_2-\tfrac{1}{2}\,m_2+\tfrac{1}{2}|j_3m_3\rangle \\ &-\{[j_1-m_1][j_2+m_2]\}^{\frac{1}{2}}\,q^{-j_1-j_2-m_1+m_2-1/2}\,{}_q\langle j_1-\tfrac{1}{2}\,m_1+\tfrac{1}{2}\,j_2-\tfrac{1}{2}\,m_2-\tfrac{1}{2}|j_3m_3\rangle\Big\} \quad (4.33)\end{aligned}$$

Iterating k times gives the relation

$$\begin{aligned}{}_q\langle j_1m_2j_2m_2|j_3m_3\rangle &= \sum_{r=0}^{k}(-)^r\frac{[k]!}{[k-r]![r]!}q^{\{(k-2r)(j_1+j_2+1)+k(-m_1+m_2)\}/2} \\ &\times\left\{\frac{[j_1+j_2-j_3-k]![j_1+j_2+j_3+1-k]!}{[j_1+j_2-j_3]![j-1+j_2+j_3+1]![j_1+m_1-k+r]!}\right. \\ &\times\left.\frac{[j_1+m_1]![j_2-m_2]![j_1-m_1]![j_2+m_2]!}{[j_2-m_2-k+r]![j_1-m_1-r]![j_2+m_2-r]!}\right\}^{\frac{1}{2}} \\ &\times\ {}_q\langle\tfrac{2j_1-k}{2}\,\tfrac{2m_1-k-2r}{2}\,\tfrac{2j_2-k}{2}\,\tfrac{2m_2+k-2r}{2}|j_3m_3\rangle \quad (4.34)\end{aligned}$$

From equation (4.34), we now obtain an expression for the stretched coupling coefficient ${}_q\langle l_1{+}l_2\, n_1\, l_1 n_2 | l_2 n_3\rangle$. With $k = 2l_1$ the only non-zero term on the right-hand side is that for which $r = l_1 + n_2$ and the trivial vector coupling coefficient is substituted for from equation (4.11)

$$
\begin{aligned}
&{}_q\langle l_1{+}l_2\, n_1\, l_1 n_2 | l_2 n_3\rangle \\
&\quad= \left\{ \frac{[0]![2l_2+1]![l_1+l_2+n_1]![l_1-n_2]![l_1+l_2-n_1]![l_1+n_2]!}{[2l_1]![2l_1+2l_2+1]![l_2+n_1+n_2]![0]![l_2-n_1-n_2]![0]!} \right\}^{\frac{1}{2}} \\
&\quad\times \frac{(-)^{l_1+n_2}[2l_1]!}{[l_1-n_2]![l_1+n_2]!} q^{(-2n_2)(2l_1+l_2+1)+(2l_1)(-n_1+n_2)}\; {}_q\langle l_2\, n_1{+}n_2\, 00 | l_2 n_3\rangle \\
&\quad= \left\{ \frac{[2l_2+1]![2l_1]![l_1+l_2+n_1]![l_1+l_2-n_1]!}{[2l_1+2l_2+1]![l_2+n_1+n_2]![l_2-n_1-n_2]![l_1+n_2]![l_1-n_2]!} \right\}^{\frac{1}{2}} \\
&\qquad\qquad\times (-)^{l_1+n_2} q^{-l_1 n_2 - l_2 n_2 - n_1 l_1 - n_2}
\end{aligned} \tag{4.35}
$$

With $k = j_1 + j_2 - j_3$, the coefficient on the right-hand side of equation (4.34) is related by symmetry (3.14) to the stretched form given in equation (4.35). Substituting and simplifying, we obtain the general vector coupling coefficient

$$
\begin{aligned}
&{}_q\langle j_1 m_1 j_2 m_2 | j_3 m_3\rangle = q^{\{j_1(j_1+1)+j_2(j_2+1)-j_3(j_3+1)+2(j_1 j_2 + j_1 m_2 - j_2 m_1)\}/2} \\
&\quad\times \{[2j_3+1][j_1+m_1]![j_1-m_1]![j_2+m_2]![j_2-m_2]![j_3+m_3]![j_3-m_3]!\}^{\frac{1}{2}} \\
&\quad\times \left\{ \frac{[j_1+j_2-j_3]![j_1-j_2+j_3]![-j_1+j_2+j_3]!}{[j_1+j_2+j_3+1]!} \right\}^{\frac{1}{2}} \sum_r \left\{ \frac{(-)^r q^{-r(j_1+j_2+j_3+1)}}{[r]![j_1+j_2-j_3-r]!} \right. \\
&\quad\times \left. \frac{1}{[j_1-m_1-r]![j_2+m_2-r]![j_3-j_2+m_1+r]![j_3-j_1-m_2+r]!} \right\}
\end{aligned} \tag{4.36}
$$

4.7 Primitive R-matrices

The non-zero primitive R-matrices are easily computed from the primitive coupling coefficient given in equation (4.32) and their expression in terms of coupling coefficients, equation (3.27). We have

$$
\begin{aligned}
\left(R_q^{\frac{1}{2} j_2}\right)_{\frac{1}{2}\, m}^{\frac{1}{2}\, m} &= \left(R_q^{\frac{1}{2} j_2}\right)_{-\frac{1}{2}\, -m}^{-\frac{1}{2}\, -m} \\
&= q^{\{c(\frac{1}{2})+c(j_2)-c(j_2+\frac{1}{2})\}}\; {}_q\langle \tfrac{1}{2}\tfrac{1}{2}\, j_2 m | j_2{+}\tfrac{1}{2}\, m{+}\tfrac{1}{2}\rangle\; {}_{\frac{1}{q}}\langle \tfrac{1}{2}\tfrac{1}{2}\, j_2 m | j_2{+}\tfrac{1}{2}\, m{+}\tfrac{1}{2}\rangle \\
&\quad + q^{\{c(\frac{1}{2})+c(j_2)-c(j_2-\frac{1}{2})\}}\; {}_q\langle \tfrac{1}{2}\tfrac{1}{2}\, j_2 m | j_2{-}\tfrac{1}{2}\, m{+}\tfrac{1}{2}\rangle\; {}_{\frac{1}{q}}\langle \tfrac{1}{2}\tfrac{1}{2}\, j_2 m | j_2{-}\tfrac{1}{2}\, m{+}\tfrac{1}{2}\rangle \\
&= \frac{1}{[2j_2+1]} \left\{ q^{-j_2}[j_2+m+1] + q^{j_2+1}[j_2-m] \right\} \\
&= q^{-m}
\end{aligned} \tag{4.37}
$$

while a similar calculation gives

$$\left(R_q^{\frac{1}{2}\,j_2}\right)^{-\frac{1}{2}\quad m}_{\frac{1}{2}\quad m-1} = (q^{-1}-q)\left\{[j_2-m+1][j_2+m]\right\}^{1/2} q^{\frac{1}{2}} \tag{4.38}$$

4.8 Calculation of general R-matrices

A recursion relation for the general case can now be obtained by letting λ_1 be the primitive irrep $\frac{1}{2}$, and m_1' be $\frac{1}{2}$ in the pentagonal equation (3.30). Substituting for the primitive R-matrices and vector coupling coefficients, and changing notation (namely $j_1-\frac{1}{2}$ for λ_2, j_1 for λ_3, j_2 for λ, etc) gives the recursion relation

$$\begin{aligned}\left(R_q^{j_1\,j_2}\right)^{n-i\quad m}_{n\quad m-i}[j_1+n]^{\frac{1}{2}} &= q^{-2m+3i/2}[j_1+n-i]^{\frac{1}{2}}\left(R_q^{j_1-\frac{1}{2}\,j_2}\right)^{n-i-\frac{1}{2}\quad m}_{n-\frac{1}{2}\quad m-i}\\ &+ q^{-2j_1+i+1/2}(q^{-1}-q)\{[j_2-m+i][j_2+m-i+1][j_1-n+i]\}^{\frac{1}{2}}\\ &\times\left(R_q^{j_1-\frac{1}{2}\,j_2}\right)^{n-i+\frac{1}{2}\quad m}_{n-\frac{1}{2}\quad m-i+1}\end{aligned} \tag{4.39}$$

Iterating this expression $2j_1$ times, with the maximum value of $n=j_1$, gives R-matrices on the right-hand side of the form $\left(R_q^{0\,j_2}\right)^{2j_1-i-t\quad m}_{0\quad m-i+2j_1-t}$. The only such R-matrix which is non-zero is the trivial R-matrix for which $2j_1-i-t=0$. Substituting, we have

$$\begin{aligned}\left(R_q^{j_1\,j_2}\right)^{j_1-i\quad m}_{j_1\quad m-i} &=\\ &\left\{\frac{[2j_1]![j_2-m+i]![j_2+m]!}{[i]![2j_1-i]![j_2+m-i]![j_2-m]!}\right\}^{\frac{1}{2}}(q^{-1}-q)^i q^{\{-(2j_1-i)(2m-i)+i\}/2}\end{aligned} \tag{4.40}$$

To obtain the R-matrices for $n<j_1$, we rearrange the recursion relation (4.39), and let j_1 go to $j_1+\frac{1}{2}$, n to $n+\frac{1}{2}$ and i to $i+1$, to give

$$\begin{aligned}\left(R_q^{j_1\,j_2}\right)^{n-i\quad m}_{n\quad m-i} &= \{[j_2-m+i+1][j_2+m-i][j_1-n+i+1]\}^{-\frac{1}{2}}(q^{-1}-q)^{-1}\\ &\times\left\{[j_1+n+1]^{\frac{1}{2}}q^{-i-1+2j_1/2}\left(R_q^{j_1+\frac{1}{2}\,j_2}\right)^{n-i-\frac{1}{2}\quad m}_{n+\frac{1}{2}\quad m-i-1}\right.\\ &\left.-[j_1+n-i]^{\frac{1}{2}}q^{-2m+2i+2j_1+2/2}\left(R_q^{j_1\,j_2}\right)^{n-i-1\quad m}_{n\quad m-i-1}\right\}\end{aligned} \tag{4.41}$$

On iteration $m-i+j_2$ times we obtain

$$\begin{aligned}
\left(R_q^{j_1\,j_2}\right)^{n-i\ \ m}_{n\ \ m-i} &= \\
&(q^{-1}-q)^{-j_2-m+i}\left\{\frac{[j_2-m+i]![j_1-n+i]![j_1+n-i]![j_2+m-i]!}{[2j_2]![j_1-n+j_2+m]![j_1+n]!}\right\}^{\frac{1}{2}} \\
&\times\sum_r \frac{(-)^r}{[r]![j_2+m-i-r]!} q^{2j_1j_2+2mj_1-2ij_1-ij_2-im+i^2-j_2-m+i+2ri+j_2r-mr+2r/2} \\
&\times\left\{\frac{[j_1+n+j_2+m-i-r]!}{[j_1+n-i-r]!}\right\}^{\frac{1}{2}}\left(R_q^{j_1+\frac{1}{2}(j_2+m-i-r)\,j_2}\right)^{n-\frac{1}{2}(j_2+m+i+r)\ \ m}_{n+\frac{1}{2}(j_2+m-i-r)\ \ -j_2}
\end{aligned} \tag{4.42}$$

The R-matrices on the right-hand side are related by symmetries (3.28) and (3.29) to the R-matrix given by equation (4.40). On substituting, we have

$$\begin{aligned}
\left(R_q^{j_1\,j_2}\right)^{n-i\ \ m}_{n\ \ m-i} &= \left\{\frac{[j_1-n+i]![j_1+n-i]![j_2+m-i]!}{[j_1+n]![j_2+m]![j_2-m]![j_1-n]!}\right\}^{\frac{1}{2}}(q^{-1}-q)^i \\
\times\, q^{2j_2n+2j_1m+2j_1j_2-2ij_1-2ij_2-2nm+i^2+i} &\sum_r \frac{(-)^r q^{ri+r}[j_1+nj_2+m-i-r]!}{[r]![j_1+n-i-r]![j_2+m-i-r]!}
\end{aligned} \tag{4.43}$$

The summation can be performed by using the identity (4.8). The resulting algebraic expression may be simplified to give the general R-matrices for $su(2)_q$ as

$$\begin{aligned}
\left(R_q^{j_1\,j_2}\right)^{n-i\ \ m}_{n\ \ m-i} =&(q^{-1}-q)^i q^{\{-(2m-i)(2n-i)+i\}/2} \\
&\times\frac{1}{[i]!}\left\{\frac{[j_1+n]![j_1-n+i]![j_2-m+i]![j_2+m]!}{[j_1-n]![j_1+n-i]![j_2+m-i]![j_2-m]!}\right\}^{\frac{1}{2}}
\end{aligned} \tag{4.44}$$

which agrees with the result obtained by Nomura (1989), when differences in the definition of q are taken into account.

Chapter 5

The q-deformed algebra $su(3)_q$

The deformed algebra $su(3)_q$ is one of the few q-deformed algebras to be studied in detail. Ma (1990a) looked at the simplest two representations, $\{1\,0\}$ and $\{2\,0\}$, finding representation matrices, vector coupling coefficients and hence R-matrices. A later paper (Ma, 1990b) extends the calculations to the $\{2\,1\}$ representation. The $6j$ symbols were considered by Archer (1992), with the Racah backcoupling and Biedenharn-Elliott relations being proved diagrammatically. However, no symbols were explicitly computed. Yu (1991) looked at representations in the $su(3)_q \supset su(2)_q \times u(1)_q$ basis. Pan and Chen (1992) computed isoscalar factors for this chain and in a later paper (Pan, 1993) considered other subalgebra chains. Quesne (1992) realizes the Gelfand-Tseitlin basis in terms of q-boson operators. Dobrev (1990) and Capps (1993) also study $su(3)_q$ representations. Some work has been done on applications (Cseh, 1993; Del Sol Mesa *et al*, 1993) However, no large classes of coupling coefficients or R-matrices have been found.

In this chapter, we explicitly calculate primitive vector coupling coefficients for $su(3)_q$ where the other representations are unrestricted. These coefficients, together with the pentagonal relation are then used to compute a large class of $su(3)_q$ R-matrices. This illustrates the general procedure for $su(n)_q$.

5.1 Structure of $su(3)_q$

The $su(3)_q$ representations are labelled by their Young tableaus, $\lambda = \{h_1\, h_2\}$. They can be obtained in a basis $u(1)_q \times su(2)_q$ as $|\lambda\sigma t\tau\rangle$, where σ is a $u(1)_q$ representation corresponding to hypercharge in $su(3)$, t is the $su(2)_q$ label of isospin and τ its z-component. This choice of basis is the same as that of Ma (1990a, 1990b), Yu (1991) and Pan and Chen (1992); Pan (1993). The primitive irrepresentation, ϵ, is chosen to be $\{1\,0\}$ (or simply $\{1\}$), its conjugate being $\{1\,1\}$. There are restrictions on σ, t and τ as given below

$$\tfrac{1}{3}h_1+\tfrac{1}{3}h_2 \geq \sigma \;\; \geq -\tfrac{2}{3}h_1+\tfrac{1}{3}h_2 \tag{5.1}$$

$$\tfrac{1}{3}h_1+\tfrac{1}{3}h_2+\tfrac{1}{2}\sigma \geq t, \quad \tfrac{2}{3}h_1-\tfrac{1}{3}h_2-\tfrac{1}{2}\sigma \geq t$$

$$t \geq \tfrac{1}{3}h_1-\tfrac{2}{3}h_2+\tfrac{1}{2}\sigma, \qquad t \geq -\tfrac{1}{3}h_1+\tfrac{2}{3}h_2-\tfrac{1}{2}\sigma \tag{5.2}$$

$$t \geq \tau \;\; \geq -t \tag{5.3}$$

From equation (3.11), the trivial vector coupling coefficient is

$$ {}_q\langle\{h_1h_2\}\sigma t\tau;\{h_1h_1-h_2\}\sigma' t'\tau'|\{0\}000\rangle = \delta_{\sigma-\sigma'}\delta_{tt'}\delta_{\tau-\tau'}\frac{(-)^{t+\frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{2}{3}h_2}q^{3\sigma-2\tau/2}[2]^{\frac{1}{2}}}{[h_1-h_2+1][h_1+2][h_2+1]^{\frac{1}{2}}} \tag{5.4}$$

where the q-dimension of $\lambda=\{h_1h_2\}$ is $|\lambda|_q=[h_1-h_2+1][h_1+2][h_2+1]/[2]$ from equation (3.1) and the phase is chosen to agree with Moshinsky (1962).

The eigenvalue of the quadratic Casimir for representations $\lambda=\{h_1h_2\}$ of $su(3)_q$ is $c(\lambda)=(h_1^2-h_1h_2+h_2^2+3h_1)/3$ (Broda, 1991; Ma, 1990a).

5.2 Primitive vector coupling coefficients

The primitive vector coupling coefficients for $su(3)_q$ are calculated from the orthogonality and symmetry properties of the coefficients following the method described in section 3.8. Phases are chosen to agree with those in $su(3)$ (Moshinsky, 1962). The primitive $su(2)_q$ coefficients are factored out.

Choosing λ_2 to be the primitive representation $\{1\}$ in the vector coupling coefficient orthogonality relations, equations (3.8) and (3.7), gives five relations for the primitive coupling coefficients. Letting $\lambda_1=\{h_1h_2\}$, $i_1=\sigma t$, we have two choices for i_2, $i_2=-\frac{2}{3}0$ or $i_2=\frac{1}{3}\frac{1}{2}$, in equation (3.8) and three for λ in equation (3.7), $\lambda=\{h_1{+}1\,h_2\}$, $\lambda=\{h_1\,h_2{+}1\}$ and $\lambda=\{h_1{-}1\,h_2{-}1\}$. Firstly, considering those coefficients for which the representations are of the form $\{h0\}$, for given σ, t is fixed as $t=\frac{1}{3}h+\frac{1}{2}\sigma$. The coefficient containing $\{1\}{-}\frac{2}{3}0$ can be eliminated using symmetry (3.14) and two relations combined to give

$$\Big|\,{}_q\langle\{h0\}\sigma-\tfrac{1}{3}\,t\,\{1\,0\}\tfrac{1}{3}\,\tfrac{1}{2}|\{h{+}1\,0\}\sigma\,t{+}\tfrac{1}{2}\rangle\Big|^2=\frac{-q^{-h-3}[2]}{[h+1]}$$
$$+\frac{q^{-3}[\frac{2}{3}h+\sigma+\frac{8}{3}]}{[\frac{2}{3}h+\sigma+\frac{5}{3}]}\Big|\,{}_q\langle\{h0\}\sigma{+}\tfrac{2}{3}\,t{+}\tfrac{1}{2}\,\{1\,0\}\tfrac{1}{3}\,\tfrac{1}{2}|\{h_1{+}1\,0\}\sigma{+}1\,t{+}1\rangle\Big|^2 \tag{5.5}$$

On iterating $\frac{1}{3}h + \frac{1}{3} - \sigma$ times, the coefficient on the right-hand side has a maximum value for σ and is equal to 1 from orthogonality. Substituting we have,

$$\Big| {}_q\langle \{h0\}\sigma{-}\tfrac{1}{3}\,t\,\{1\,0\}\tfrac{1}{3}\,\tfrac{1}{2}|\{h{+}1\,0\}\sigma\,t{+}\tfrac{1}{2}\rangle\Big|^2 =$$

$$\frac{-q^{-h}[2]}{[h+1][\frac{2}{3}h+\sigma+\frac{5}{3}]}\sum_{k=0}^{\frac{1}{3}h+\frac{1}{3}+\sigma}[\tfrac{2}{3}h+\sigma+\tfrac{2}{3}+k]q^{-3k}+\frac{q^{-h-1+3\sigma}[h+2]}{[\frac{2}{3}h+\sigma+\frac{5}{3}]} \qquad (5.6)$$

Finally, using the q-number identities, equations (4.2), (4.3), (4.4) and (4.6), we obtain

$$\Big| {}_q\langle \{h0\}\sigma{-}\tfrac{1}{3}\,t\,\{1\,0\}\tfrac{1}{3}\,\tfrac{1}{2}|\{h{+}1\,0\}\sigma\,t{+}\tfrac{1}{2}\rangle\Big|^2 = q^{-\frac{1}{3}h+\sigma-\frac{1}{3}}\frac{[\frac{2}{3}h+\sigma+\frac{2}{3}]}{[h+1]} \qquad (5.7)$$

Other coefficients with $h_2 = 0$ can be found from this one. The coefficients with arbitrary h_2 and maximum t can then be obtained and finally the most general coefficients. With the $su(2)_q$ primitive coefficients factored out, we have the coefficients shown in Table 5.1

The coupling coefficients obtained here agree with those of Ma (1990a, 1990b) and Pan (1993) where there is an overlap. In the $q \to 1$ limit, the $su(3)$ coefficients of Moshinsky (1962) are recovered.

5.3 Primitive R-matrices

The primitive R-matrices for $su(3)_q$ are obtained from the primitive vector coupling coefficients found for $su(2)_q$ and $su(3)_q$, equation (4.32) and Table 5.1 respectively, using equation (3.27) which relates R-matrices to coupling coefficients. The non-zero $su(3)_q$ primitive R-matrices are given in Table 5.2. Note that the first four R-matrices can be factorized into a primitive $su(2)_q$ R-matrix and a $su(3)_q$ part $q^{-\frac{3}{2}\rho\sigma}$ where $\rho = -\frac{2}{3}$ or $\rho = \frac{1}{3}$, but this is not true for the remaining R-matrices.

5.4 Calculation of general R-matrices

The pentagonal equation, (3.30), is used as a recursion relation. One irrep, λ_2, is taken as the primitive irrep $\{1\}$. With three choices of m_2' and three choices for λ_3 there are nine different equations. We give four of these below, these will be used in the calculation of general R-matrices.

$$\left(R_q^{h_1h_2\,g_1g_2}\right)^{\sigma-i\,t-\frac{1}{2}i\,\tau-j\quad\rho\,s\,\nu}_{\sigma\,t\,\tau\quad\rho-i\,s-\frac{1}{2}i\,\nu} =$$

$$q^{\rho-\frac{1}{2}i}\left\{\frac{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+i]}{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t]}\right\}^{\frac{1}{2}}\left(R_q^{h_1-1h_2\,g_1g_2}\right)^{\sigma+\frac{2}{3}-i\,t-\frac{1}{2}i\,\tau-j\quad\rho\,s\,\nu}_{\sigma+\frac{2}{3}\,t\,\tau\quad\rho-i\,s-\frac{1}{2}i\nu-j} \qquad (5.8)$$

Table 5.1: Primitive $su(3)_q$ coupling coefficients

${}_q\langle\{h_1h_2\}\sigma t;\{1\}{-\tfrac{2}{3}}0\|\{h_1{+}1h_2\}\sigma{-}\tfrac{2}{3}t\rangle$	$q^{\frac{1}{3}h_1-\frac{1}{6}h_2+\frac{1}{2}\sigma}\left\{\dfrac{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+1][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma+t+2]}{[h_1-h_2+1][h_1+2]}\right\}^{\frac{1}{2}}$
${}_q\langle\{h_1h_2\}\sigma t;\{1\}{-\tfrac{2}{3}}0\|\{h_1h_2{+}1\}\sigma{-}\tfrac{2}{3}t\rangle$	$q^{-\frac{1}{6}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-\frac{1}{2}}\left\{\dfrac{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t][-\frac{1}{3}h_1+\frac{2}{3}h_2-\frac{1}{2}\sigma+t+1]}{[h_1-h_2+1][h_2+1]}\right\}^{\frac{1}{2}}$
${}_q\langle\{h_1h_2\}\sigma t;\{1\}\tfrac{1}{3}\tfrac{1}{2}\|\{h_1{+}1h_2\}\sigma{+}\tfrac{1}{3}t{+}\tfrac{1}{2}\rangle$	$q^{-\frac{1}{3}h_1+\frac{1}{6}h_2+\frac{1}{4}\sigma+\frac{1}{2}t}\left\{\dfrac{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t+1][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma+t+2][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+2]}{[2t+2][h_1-h_2+1][h_1+2]}\right\}^{\frac{1}{2}}$
${}_q\langle\{h_1h_2\}\sigma t;\{1\}\tfrac{1}{3}\tfrac{1}{2}\|\{h_1h_2{+}1\}\sigma{+}\tfrac{1}{3}t{+}\tfrac{1}{2}\rangle$	$-q^{\frac{1}{6}h_1-\frac{1}{3}h_2+\frac{1}{4}\sigma+\frac{1}{2}t+\frac{1}{2}}\left\{\dfrac{[-\frac{1}{3}h_1+\frac{2}{3}h_2-\frac{1}{2}\sigma+t+1][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+2][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+2]}{[2t+2][h_1-h_2+1][h_2+1]}\right\}^{\frac{1}{2}}$
${}_q\langle\{h_1h_2\}\sigma t;\{1\}\tfrac{1}{3}\tfrac{1}{2}\|\{h_1{+}1h_2\}\sigma{+}\tfrac{1}{3}t{-}\tfrac{1}{2}\rangle$	$q^{-\frac{1}{3}h_1+\frac{1}{6}h_2+\frac{1}{4}\sigma-\frac{1}{2}t-\frac{1}{2}}\left\{\dfrac{[-\frac{1}{3}h_1+\frac{2}{3}h_2-\frac{1}{2}\sigma+t][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+1][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-t+1]}{[2t][h_1-h_2+1][h_1+2]}\right\}^{\frac{1}{2}}$
${}_q\langle\{h_1h_2\}\sigma t;\{1\}\tfrac{1}{3}\tfrac{1}{2}\|\{h_1h_2{+}1\}\sigma{+}\tfrac{1}{3}t{-}\tfrac{1}{2}\rangle$	$q^{\frac{1}{6}h_1-\frac{1}{3}h_2+\frac{1}{4}\sigma-\frac{1}{2}t}\left\{\dfrac{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma+t+1][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-t+1]}{[2t][h_1-h_2+1][h_2+1]}\right\}^{\frac{1}{2}}$

Table 5.2: Primitive $su(3)_q$ R-matrices

$\left(R_q^{h_1 h_2\, 10}\right)^{\sigma\, t\, \tau\, -\frac{2}{3}\, 0\, 0}_{\sigma\, t\, \tau\, -\frac{2}{3}\, 0\, 0}$	q^{σ}
$\left(R_q^{h_1 h_2\, 10}\right)^{\sigma\, t\, \tau\, \frac{1}{3}\, \frac{1}{2}\, \frac{1}{2}}_{\sigma\, t\, \tau\, \frac{1}{3}\, \frac{1}{2}\, \frac{1}{2}}$	$q^{-\frac{1}{2}\sigma-\tau}$
$\left(R_q^{h_1 h_2\, 10}\right)^{\sigma\, t\, \tau\, \frac{1}{3}\, \frac{1}{2}\, -\frac{1}{2}}_{\sigma\, t\, \tau\, \frac{1}{3}\, \frac{1}{2}\, -\frac{1}{2}}$	$q^{-\frac{1}{2}\sigma+\tau}$
$\left(R_q^{h_1 h_2\, 10}\right)^{\sigma\, t\, \tau-1\, \frac{1}{3}\, \frac{1}{2}\, \frac{1}{2}}_{\sigma\, t\, \tau\, \frac{1}{3}\, \frac{1}{2}\, -\frac{1}{2}}$	$q^{-\frac{1}{2}\sigma+\frac{1}{2}}(q^{-1}-q)\{[t+\tau][t-\tau+1]\}^{\frac{1}{2}}$
$\left(R_q^{h_1 h_2\, 10}\right)^{\sigma-1\, t+\frac{1}{2}\, \tau-\frac{1}{2}\, \frac{1}{3}\, \frac{1}{2}\, \frac{1}{2}}_{\sigma\, t\, \tau\, -\frac{2}{3}\, 0\, 0}$	$q^{\frac{1}{4}\sigma+\frac{1}{2}\tau+t+\frac{3}{2}}(q-q^{-1})\left\{\frac{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-t][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma+t+2][-\frac{1}{3}h_1+\frac{2}{3}h_2-\frac{1}{2}\sigma+t+1][t-\tau+1]}{[2t+2][2t+1]}\right\}^{\frac{1}{2}}$
$\left(R_q^{h_1 h_2\, 10}\right)^{\sigma-1\, t-\frac{1}{2}\, \tau-\frac{1}{2}\, \frac{1}{3}\, \frac{1}{2}\, \frac{1}{2}}_{\sigma\, t\, \tau\, -\frac{2}{3}\, 0\, 0}$	$q^{\frac{1}{4}\sigma+\frac{1}{2}\tau-t+\frac{1}{2}}(q^{-1}-q)\left\{\frac{[t+\tau][\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+1][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+1]}{[2t][2t+1]}\right\}^{\frac{1}{2}}$
$\left(R_q^{h_1 h_2\, 10}\right)^{\sigma-1\, t+\frac{1}{2}\, \tau+\frac{1}{2}\, \frac{1}{3}\, \frac{1}{2}\, -\frac{1}{2}}_{\sigma\, t\, \tau\, -\frac{2}{3}\, 0\, 0}$	$q^{\frac{1}{4}\sigma+\frac{1}{2}\tau+\frac{1}{2}}(q^{-1}-q)\left\{\frac{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-t][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma+t+2][-\frac{1}{3}h_1+\frac{2}{3}h_2-\frac{1}{2}\sigma+t+1][t+\tau+1]}{[2t+2][2t+1]}\right\}^{\frac{1}{2}}$
$\left(R_q^{h_1 h_2\, 10}\right)^{\sigma-1\, t-\frac{1}{2}\, \tau+\frac{1}{2}\, \frac{1}{3}\, \frac{1}{2}\, -\frac{1}{2}}_{\sigma\, t\, \tau\, -\frac{2}{3}\, 0\, 0}$	$q^{\frac{1}{4}\sigma+\frac{1}{2}\tau+\frac{1}{2}}(q^{-1}-q)\left\{\frac{[t-\tau][\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+1][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+1]}{[2t][2t+1]}\right\}^{\frac{1}{2}}$

$$
\begin{aligned}
&\left(R_q^{h_1 h_2\, g_1 g_2}\right)^{\sigma-i\; t-\frac{1}{2}i\; \tau-j \qquad \rho\; s\; \nu}_{\sigma\; t\; \tau \qquad \rho-i\; s-\frac{1}{2}i\; \nu} = \\
&q^{\rho-\frac{1}{2}i}\left\{\frac{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t-i+1]}{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t+1]}\right\}^{\frac{1}{2}}\left(R_q^{h_1 h_2-1\, g_1 g_2}\right)^{\sigma+\frac{2}{3}-i\; t-\frac{1}{2}i\; \tau-j \qquad \rho\; s\; \nu}_{\sigma+\frac{2}{3}\; t\; \tau \qquad \rho-i\; s-\frac{1}{2}i\nu-j}
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
&\left(R_q^{h_1 h_2\, g_1 g_2}\right)^{\sigma-i\; t-\frac{1}{2}i\; \tau-j \qquad \rho\; s\; \nu}_{\sigma\; t\; \tau \qquad \rho-i\; s-\frac{1}{2}i\; \nu} = \\
&\quad q^{\rho-\frac{1}{2}i}\left\{\frac{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t-i+2]}{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+2]}\right\}^{\frac{1}{2}}\left(R_q^{h_1+1 h_2+1\, g_1 g_2}\right)^{\sigma+\frac{2}{3}-i\; t-\frac{1}{2}i\; \tau-j \qquad \rho\; s\; \nu}_{\sigma+\frac{2}{3}\; t\; \tau \qquad \rho-i\; s-\frac{1}{2}i\nu-j}
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
&\left(R_q^{h_1+1 0\, g_1 0}\right)^{\sigma+\frac{1}{3}-i\; t+\frac{1}{2}-\frac{1}{2}i\; \tau-j \qquad \rho\; s\; \nu}_{\sigma+\frac{1}{3}\; t+\frac{1}{2}\; \tau \qquad \rho-i\; s-\frac{1}{2}i\; \nu-j}\; q^{-\frac{1}{3}h_1+\frac{1}{4}\sigma+\frac{1}{2}\tau-\frac{1}{4}}[\tfrac{1}{2}\sigma+\tfrac{1}{3}h_1+\tau+\tfrac{1}{2}]^{\frac{1}{2}} \\
&\quad =\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i+1\; t-\frac{1}{2}i+\frac{1}{2}\; \tau-j\; \rho-1\; s-\frac{1}{2}\; \nu-\frac{1}{2}}_{\sigma\; t\; \tau-\frac{1}{2} \qquad \rho-i\; s-\frac{1}{2}i\; \nu}\; (q^{-1}-q)q^{\frac{1}{3}h_1+\frac{1}{2}\sigma-\frac{1}{2}i-\frac{1}{4}\rho+\frac{1}{2}\nu-\frac{1}{3}g_1+1} \\
&\qquad \times\left\{[\tfrac{1}{2}\rho+\tfrac{1}{3}g_1+\nu][\tfrac{1}{3}h_1-\sigma+i][\tfrac{1}{3}g_1-\rho+1]\right\}^{\frac{1}{2}} \\
&\quad +\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\; t-\frac{1}{2}i\; \tau-\frac{1}{2}-j \qquad \rho\; s\; \nu}_{\sigma\; t\; \tau-\frac{1}{2} \qquad \rho-i\; s-\frac{1}{2}i\; \nu-j}\; q^{-\frac{1}{3}h_1+\frac{1}{4}\sigma-\frac{1}{4}i+\frac{1}{2}\tau-\frac{1}{2}j-\frac{1}{2}\rho-\nu-\frac{1}{4}} \\
&\qquad \times[\tfrac{1}{2}\sigma+\tfrac{1}{3}h_1-\tfrac{1}{2}i+\tau-j+\tfrac{1}{2}]^{\frac{1}{2}} \\
&\quad +\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\; t-\frac{1}{2}i\; \tau+\frac{1}{2}-j \qquad \rho\; s\; \nu-1}_{\sigma\; t\; \tau-\frac{1}{2} \qquad \rho-i\; s-\frac{1}{2}i\; \nu-j}\; q^{\frac{3}{4}\sigma-\frac{3}{4}i+\frac{1}{2}\tau-\frac{1}{2}j-\frac{1}{2}\rho+\frac{1}{4}} \\
&\qquad \times\left\{[\tfrac{1}{2}\rho+\tfrac{1}{3}g_1+\nu][\tfrac{1}{2}\rho+\tfrac{1}{3}g_1-\nu+1][\tfrac{1}{2}\sigma+\tfrac{1}{3}h_1-\tfrac{1}{2}i-\tau+j+\tfrac{1}{2}]\right\}^{\frac{1}{2}}
\end{aligned}
\tag{5.11}
$$

where in all the above we have chosen $t-\frac{1}{2}i$ and $s-\frac{1}{2}i$ rather than the more general $t-l$ and $s-m$. In addition, in equation (5.11) we have taken $h_2 = g_2 = 0$ thus restricting t and s to be $t=\frac{1}{2}\sigma+\frac{1}{3}h_1$ and $s=\frac{1}{2}\rho+\frac{1}{3}g_1$ which eliminates some terms.

With equation (5.11), taking $\sigma = \frac{1}{3}h_1$, $\tau = t = \frac{1}{2}h_1$ and $h_1 \to h_1 - 1$ and rearranging we have

$$
\begin{aligned}
&\left(R_q^{h_1 0\, g_1 0}\right)^{\frac{1}{3}h_1-i\;\frac{1}{2}h_1-\frac{1}{2}i\;\frac{1}{2}h_1-j\qquad \rho\;\frac{1}{3}g_1+\frac{1}{2}\rho\;\nu}_{\sigma\;\frac{1}{2}h_1\;\frac{1}{2}h_1\qquad \rho-i\;\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\nu-j} = \\
&\quad \left(R_q^{h_1-1 0\, g_1 0}\right)^{\frac{1}{3}h_1+\frac{2}{3}-i\;\frac{1}{2}h_1-\frac{1}{2}i\;\frac{1}{2}h_1-j\;\rho-1\;\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}\;\nu-\frac{1}{2}}_{\frac{1}{3}h_1-\frac{1}{3}\;\frac{1}{2}h_1-\frac{1}{2}\;\frac{1}{2}h_1-\frac{1}{2}\quad \rho-i\;\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\nu-j}\left(q^{-1}-q\right) \\
&\quad \times q^{\frac{1}{2}h_1-\frac{1}{2}i-\frac{1}{3}g_1-\frac{1}{4}\rho+\frac{1}{2}\nu+\frac{1}{2}}\left\{\frac{[\frac{1}{3}g_1+\frac{1}{2}\rho+\nu][\frac{1}{3}g_1-\rho+1][i]}{[h_1]}\right\}^{\frac{1}{2}} \\
&\quad +\left(R_q^{h_1-1 0\, g_1 0}\right)^{\frac{1}{3}h_1-\frac{1}{3}-i\;\frac{1}{2}h_1-\frac{1}{2}i-\frac{1}{2}\;\frac{1}{2}h_1-\frac{1}{2}-j\qquad \rho\;\frac{1}{3}g_1+\frac{1}{2}\rho\;\nu}_{\frac{1}{3}h_1-\frac{1}{3}\;\frac{1}{2}h_1-\frac{1}{2}\;\frac{1}{2}h_1-\frac{1}{2}\qquad \frac{1}{3}g_1-i\;\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\nu-j} \\
&\quad \times q^{-\frac{1}{4}i-\frac{1}{2}j-\frac{1}{2}\rho-\nu}\left\{\frac{[h_1-\frac{1}{2}i-j]}{[h_1]}\right\}^{\frac{1}{2}} \\
&\quad +\left(R_q^{h_1-1 0\, g_1 0}\right)^{\frac{1}{3}h_1-\frac{1}{3}-i\;\frac{1}{2}h_1-\frac{1}{2}i-\frac{1}{2}\;\frac{1}{2}h_1+\frac{1}{2}-j\quad \rho\;\frac{1}{3}g_1+\frac{1}{2}\rho\;\nu-1}_{\frac{1}{3}h_1-\frac{1}{3}\;\frac{1}{2}h_1-\frac{1}{2}\;\frac{1}{2}h_1-\frac{1}{2}\qquad \rho-i\;\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\nu-j}\left(q^{-1}-q\right) \\
&\quad \times q^{\frac{1}{2}h_1-\frac{3}{4}i-\frac{1}{2}j-\frac{1}{2}\rho+\frac{1}{2}}\left\{\frac{[\frac{1}{3}g_1+\frac{1}{2}\rho+\nu][\frac{1}{3}g_1+\frac{1}{2}\rho-\nu+1][j-\frac{1}{2}i]}{[h_1]}\right\}^{\frac{1}{2}}
\end{aligned}
\tag{5.12}
$$

On iterating h_1 times we have

$$
\begin{aligned}
&\left(R_q^{h_1 0\, g_1 0}\right)^{\frac{1}{3}h_1-i\;\frac{1}{2}h_1-\frac{1}{2}i\;\frac{1}{2}h_1-j\qquad \rho\;\frac{1}{3}g_1+\frac{1}{2}\rho\;\nu}_{\frac{1}{3}h_1\;\frac{1}{2}h_1\;\frac{1}{2}h_1\qquad \rho-i\;\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\nu-j} = \sum_{b,c}\frac{(q^{-1}-q)^{h_1-b+c}}{[h_1-b]![b-c]![c]!} \\
&\times q^{b(-\frac{1}{4}i+\frac{1}{2}j+\frac{1}{4}g_1+\frac{1}{2}\nu+h_1+\frac{1}{2}-\frac{1}{2}b)+c(-\frac{1}{2}i+\nu+h_1+\frac{1}{2}-\frac{1}{2}c-\frac{1}{2}b)-\frac{1}{4}h_1 i-\frac{1}{2}h_1 j-\frac{1}{6}h_1 g_1-h_1\nu} \\
&\times\left\{\frac{[i]![j-\frac{1}{2}i]![h_1]![h_1-\frac{1}{2}i-j]!}{[h_1-\frac{1}{2}i-j-b+c]![i-h_1+b]![j-\frac{1}{2}i-c]!}\right. \\
&\qquad\left.\times\frac{[\frac{1}{3}g_1+\frac{1}{2}\rho+\nu]![\frac{1}{3}g_1-\rho+h_1-b]![\frac{1}{3}g_1+\frac{1}{2}\rho-\nu+c]!}{[\frac{1}{3}g_1+\frac{1}{2}\rho+\nu-h_1+b-c]![\frac{1}{3}g_1-\rho]![\frac{1}{3}g_1+\frac{1}{2}\rho-\nu]!}\right\}^{\frac{1}{2}} \\
&\times\left(R_q^{00\, g_1 0}\right)^{-i+h_1-b\;-\frac{1}{2}i+\frac{1}{2}h_1-\frac{1}{2}b\;-j+c+\frac{1}{2}h_1-\frac{1}{2}b\;\rho-h_1+b\;\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}h_1+\frac{1}{2}b\;\nu-c-\frac{1}{2}h_1+\frac{1}{2}b}_{0\;0\;0\qquad \rho-i\;\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\nu-j}
\end{aligned}
\tag{5.13}
$$

The only non-zero R-matrix on the right-hand side is that for which $b=h_1-i$ and $c=j-\frac{1}{2}i$. Substituting for the trivial R-matrix we have

$$
\begin{aligned}
&\left(R_q^{h_1 0\, g_1 0}\right)^{\frac{1}{3}h_1-i\;\frac{1}{2}h_1-\frac{1}{2}i\;\frac{1}{2}h_1-j\qquad \rho\;\frac{1}{3}g_1+\frac{1}{2}\rho\;\nu}_{\frac{1}{3}h_1\;\frac{1}{2}h_1\;\frac{1}{2}h_1\qquad \rho-i\;\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\nu-j} = \\
&\left\{\frac{[h_1]![\frac{1}{3}g_1-\rho+i]![\frac{1}{3}g_1+\frac{1}{2}\rho+\nu]![\frac{1}{3}g_1+\frac{1}{2}\rho-\nu+j-\frac{1}{2}i]!}{[i]![j-\frac{1}{2}i]![h_1-\frac{1}{2}i-j]![\frac{1}{3}g_1+\frac{1}{2}\rho+\nu-j-\frac{1}{2}i]![\frac{1}{3}g_1-\rho]![\frac{1}{3}g_1+\frac{1}{2}\rho-\nu]!}\right\}^{\frac{1}{2}} \\
&\qquad\qquad\times(q^{-1}-q)^{j+\frac{1}{2}i}q^{-\frac{1}{6}g_1h_1-h_1\nu+\frac{1}{4}ig_1+\nu j+\frac{1}{2}h_1 j+\frac{1}{4}h_1 i+\frac{1}{4}i+\frac{1}{2}j-\frac{1}{2}j^2-\frac{3}{8}i^2}
\end{aligned}
\tag{5.14}
$$

To obtain the R-matrix with general σ we use equation (5.4)

$$\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-j\qquad \rho\;\; \frac{1}{2}g_1\;\; \nu}_{\sigma\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\qquad \rho-i\;\; \frac{1}{3}g_1+\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\; \nu-j} = \left\{\frac{[\frac{1}{3}h_1-\sigma+i]}{[\frac{1}{3}h_1-\sigma]}\right\}^{\frac{1}{2}}$$

$$\times\, q^{\rho-\frac{1}{2}i}\left(R_q^{h_1-1 0\, g_1 0}\right)^{\sigma+\frac{2}{3}-i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-j\qquad \rho\;\; \frac{1}{2}g_1\;\; \nu}_{\sigma+\frac{2}{3}\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\qquad \rho-i\;\; \frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\; \nu-j} \tag{5.15}$$

and iterating $\frac{1}{3}h_1-\sigma$ times

$$\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-j\qquad \rho\;\; \frac{1}{3}g_1+\frac{1}{2}\rho\;\; \nu}_{\sigma\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\qquad \rho-i\;\; \frac{1}{3}g_1+\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\; \nu-j} = \left\{\frac{[\frac{1}{3}h_1-\sigma+i]!}{[\frac{1}{3}h_1-\sigma]![i]!}\right\}^{\frac{1}{2}}$$

$$\times q^{\frac{1}{3}\rho h_1-\rho\sigma-\frac{1}{6}ih_1+\frac{1}{2}i\sigma}\left(R_q^{\frac{2}{3}h+\sigma 0\, g_1 0}\right)^{\frac{1}{3}\sigma+\frac{2}{9}h_1-i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-j\qquad \rho\;\; \frac{1}{3}g_1+\frac{1}{2}\rho\;\; \nu}_{\frac{1}{3}\sigma+\frac{2}{9}h_1\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\qquad \rho-i\;\; \frac{1}{3}g_1+\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\; \nu-j} \tag{5.16}$$

and substituting for the R-matrix on the right-hand side from equation (5.14)

$$\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-j\qquad \rho\;\; \frac{1}{3}g_1+\frac{1}{2}\rho\;\; \nu}_{\sigma\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\qquad \rho-i\;\; \frac{1}{3}g_1+\frac{1}{3}g_1+\frac{1}{2}\rho-\frac{1}{2}i\;\; \nu-j} = (q^{-1}-q)^{j+\frac{1}{2}i}$$

$$\times\left\{\frac{[\frac{1}{3}h_1-\sigma+i]![\frac{2}{3}h_1+\sigma]![\frac{1}{3}g_1-\rho+i]![\frac{1}{3}g_1+\frac{1}{2}\rho+\nu]![\frac{1}{3}g_1+\frac{1}{2}\rho-\nu+j-\frac{1}{2}i]!}{[\frac{1}{3}h_1-\sigma]![\frac{2}{3}h_1+\sigma-j-\frac{1}{2}i]![\frac{1}{3}g_1-\rho]![\frac{1}{3}g_1+\frac{1}{2}\rho+\nu-j-\frac{1}{2}i]![\frac{1}{3}g_1+\frac{1}{2}\rho-\nu]!}\right\}^{\frac{1}{2}}$$

$$\times q^{\frac{3}{4}\sigma i-\frac{3}{2}\sigma\rho-\frac{1}{2}ij+\frac{1}{3}h_1 j+\frac{1}{2}\sigma j+\frac{1}{4}\rho i-\frac{1}{3}g_1 i-\frac{2}{3}h_1\nu-\sigma\nu+\nu i+\nu j+\frac{1}{4}i+\frac{1}{2}j-\frac{1}{2}j^2-\frac{1}{8}i^2}\frac{1}{[i]!\{[j-\frac{1}{2}i]!\}^{\frac{1}{2}}} \tag{5.17}$$

To obtain R-matrices with general $\tau\neq t$ we use equation (5.11) for a second time. On rearranging, moving the last term on the right-hand side to the left and adjusting the variables we have

$$\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\;\; \tau-j\qquad \rho\;\; \frac{1}{2}\rho+\frac{1}{3}g_1\;\; \nu}_{\sigma\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\;\; \tau\qquad \rho-i\;\; \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\; \nu-j} =$$

$$\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i+1\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i+\frac{1}{2}\;\; \tau-j-\frac{1}{2}\;\; \rho-1\;\; \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}\;\; \nu+\frac{1}{2}}_{\sigma\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\;\; \tau\qquad \rho-i\;\; \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\; \nu-j}$$

$$\times\, q^{\frac{1}{4}\rho-\frac{1}{2}\tau+\frac{1}{2}j+\frac{1}{2}\nu-\frac{1}{3}g_1}\left\{\frac{[\frac{1}{3}g_1-\rho+1][\frac{1}{3}h_1-\sigma+i]}{[\frac{1}{2}\rho+\frac{1}{3}g_1-\nu][\frac{1}{2}\sigma+\frac{1}{3}h_1-\tau+j-\frac{1}{2}i+1]}\right\}^{\frac{1}{2}}$$

$$+\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\;\; \tau-j-1\qquad \rho\;\; \frac{1}{2}\rho+\frac{1}{3}g_1\;\; \nu+1}_{\sigma\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1\;\; \tau\qquad \rho-i\;\; \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\; \nu-j}(q^{-1}-q)^{-1}$$

$$\times\, q^{-\nu}\left\{\frac{[\frac{1}{2}\sigma+\frac{1}{3}h_1+\tau-j-\frac{1}{2}i]}{[\frac{1}{2}\rho+\frac{1}{3}g_1+\nu+1][\frac{1}{2}\rho+\frac{1}{3}g_1-\nu][\frac{1}{2}\sigma+\frac{1}{3}h_1-\tau+j-\frac{1}{2}i+1]}\right\}^{\frac{1}{2}}$$

$$-\left(R_q^{h_1+1 0\, g_1 0}\right)^{\sigma+\frac{1}{3}-i\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{2}-\frac{1}{2}i\;\; \tau-j-\frac{1}{2}\qquad \rho\;\; \frac{1}{2}\rho+\frac{1}{3}g_1\;\; \nu+1}_{\sigma+\frac{1}{3}\;\; \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{2}\;\; \tau+\frac{1}{2}\qquad \rho-i\;\; \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\; \nu-j}(q^{-1}-q)^{-1}$$

$$\times\, q^{i+\frac{1}{2}j+\frac{1}{2}\rho}\left\{\frac{[\frac{1}{2}\sigma+\frac{1}{3}h_1+\tau+1]}{[\frac{1}{2}\rho+\frac{1}{3}g_1+\nu+1][\frac{1}{2}\rho+\frac{1}{3}g_1-\nu][\frac{1}{2}\sigma+\frac{1}{3}h_1-\tau+j-\frac{1}{2}i+1]}\right\}^{\frac{1}{2}} \tag{5.18}$$

On iterating $a = \frac{1}{2}\rho + \frac{1}{3}g_1 - \nu$ times

$$
\begin{aligned}
&\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\;\frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\;\tau-j\qquad\rho\;\frac{1}{2}\rho+\frac{1}{3}g_1\;\nu}_{\sigma\;\frac{1}{2}\sigma+\frac{1}{3}h_1\;\tau\qquad\rho-i\;\frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\nu-j} = \sum_{b,c} \frac{(-)^c (q^{-1}-q)^{-b}[\frac{1}{2}\rho+\frac{1}{3}g_1-\nu]!}{[\frac{1}{2}\rho+\frac{1}{3}g_1-\nu-b]![b-c]![c]!} \\
&\times q^{a(\frac{1}{3}h_1-\frac{1}{4}\sigma+\frac{1}{4}i-\frac{1}{2}\tau+\frac{1}{4}\rho+\frac{1}{2}j+\frac{1}{2}\nu-\frac{1}{3}g_1+\frac{3}{2})+b(-\frac{2}{3}h_1-\frac{1}{4}\sigma+\frac{1}{4}i-\frac{1}{4}\rho+\frac{1}{2}\tau-\frac{1}{2}j-\frac{3}{2}\nu+\frac{1}{3}g_1-\frac{5}{2}-\frac{1}{2}b)} \\
&\times q^{c(\frac{1}{4}i+\frac{1}{2}\rho+\frac{1}{2}j+\nu+1+\frac{1}{2}b)} \left\{ \frac{[\frac{1}{2}\rho+\frac{1}{3}g_1+\nu]!}{[\frac{1}{3}g_1-\rho]![\frac{1}{2}\rho+\frac{1}{3}g_1+\nu+b]!} \right. \\
&\times \frac{[\frac{1}{3}h_1-\sigma+i]![\frac{2}{3}g_1-\frac{1}{2}\rho-\nu-b]![\frac{1}{2}\sigma+\frac{1}{3}h_1-\tau+j-\frac{1}{2}i]!}{[\frac{1}{3}h_1-\sigma+i-\frac{1}{2}\rho-\frac{1}{3}g_1+\nu+b]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu]![\frac{1}{2}\sigma+\frac{1}{3}h_1+\tau]!} \\
&\times \left. \frac{[\frac{1}{2}\sigma+\frac{1}{3}h_1+\tau-j-\frac{1}{2}i]![\frac{1}{2}\sigma+\frac{1}{3}h_1+\tau+c]!}{[\frac{1}{2}\sigma+\frac{1}{3}h_1-\tau+j-\frac{1}{2}i+\frac{1}{3}g_1+\frac{1}{2}\rho-\nu]![\frac{1}{2}\rho+\frac{1}{3}h_1+\tau-j-\frac{1}{2}i-b+c]!} \right\}^{\frac{1}{2}} \\
&\times \left(R_q^{h_1+c 0\, g_1 0}\right)^{\sigma+\frac{1}{3}c-i+a-b\;t+\frac{1}{2}c-\frac{1}{2}i+\frac{1}{2}a-\frac{1}{2}b\;\tau+\frac{1}{2}c-j-\frac{1}{2}a-\frac{1}{2}b\;\rho-a+b\;s-\frac{1}{2}a+\frac{1}{2}b\;\frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}a+\frac{1}{2}b}_{\sigma+\frac{1}{3}c\;t+\frac{1}{2}c\;\tau+\frac{1}{2}c\qquad\qquad\rho-i\;s-\frac{1}{2}i\;\nu-j}
\end{aligned}
\tag{5.19}
$$

where $t = \frac{1}{2}\sigma + \frac{1}{3}h_1$, $s = \frac{1}{2}\rho + \frac{1}{3}g_1$. Using crossing symmetry, equation (3.28), to substitute for the R-matrix on the right-hand side from equation (5.17) we obtain

$$
\begin{aligned}
&\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\;\frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\;\tau-j\qquad\rho\;\frac{1}{2}\rho+\frac{1}{3}g_1\;\nu}_{\sigma\;\frac{1}{2}\sigma+\frac{1}{3}h_1\;\tau\qquad\rho-i\;\frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\nu-j} = (q^{-1}-q)^{j+\frac{1}{2}i} q^{\frac{1}{3}h_1(-i+\rho-2\nu+\frac{2}{3}g_1)} \\
&\times q^{\frac{1}{3}g_1(-\frac{1}{2}i+j+\sigma-2\tau)+\sigma(-\rho-\nu+\frac{1}{4}i)+\rho(\frac{1}{2}i+\frac{1}{2}j-\tau)+\frac{1}{2}\nu i-\frac{1}{2}ij+\tau i+\tau j+\frac{1}{4}i+\frac{1}{2}j-\frac{1}{2}j^2-\frac{1}{8}+\frac{3}{4}\rho+\frac{1}{2}g_1-\frac{3}{2}\nu} \\
&\times \left\{ \frac{[\frac{1}{3}g_1-\rho+i]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu]![\frac{1}{2}\rho+\frac{1}{3}g_1+\nu]!}{[\frac{1}{2}\rho+\frac{1}{3}g_1-\nu+j-\frac{1}{2}i]![\frac{1}{2}\rho+\frac{1}{3}g_1+\nu-j-\frac{1}{2}i]![\frac{1}{3}g_1-\rho]!} \right. \\
&\qquad \times \left. \frac{[\frac{1}{2}\sigma+\frac{1}{3}h_1-\tau+j-\frac{1}{2}i]![\frac{1}{2}\sigma+\frac{1}{3}h_1+\tau-j-\frac{1}{2}i]![\frac{1}{3}h_1-\sigma+i]!}{[\frac{1}{3}h_1-\sigma]![\frac{1}{2}\sigma+\frac{1}{3}h_1-\tau]![\frac{1}{2}\sigma+\frac{1}{3}h_1+\tau]!} \right\}^{\frac{1}{2}} \\
&\times \sum_{b,c} (-)^c q^{\frac{1}{2}bi-\frac{3}{2}b\sigma-bj+\frac{1}{2}ic-h_1 b+b\tau+cj-2b+c} [\tfrac{1}{2}\sigma+\tfrac{1}{3}h_1+\tau+c]! \left\{ [\tfrac{1}{2}\rho+\tfrac{1}{3}g_1-\nu-b]! \right. \\
&\qquad \times \left. [\tfrac{1}{2}\sigma+\tfrac{1}{3}h_1+\tau-j-\tfrac{1}{2}i-b+c]![i-\tfrac{1}{2}\rho-\tfrac{1}{3}g_1+\nu+b]![b-c]![c]! \right\}^{-1}
\end{aligned}
\tag{5.20}
$$

Finally, using equation (4.8) to sum over c we have

$$
\begin{aligned}
\left(R_q^{h_1 0\, g_1 0}\right)^{\sigma-i\ \frac{1}{2}\sigma+\frac{1}{3}h_1-\frac{1}{2}i\ \tau-j \qquad \rho\ \frac{1}{2}\rho+\frac{1}{3}g_1\ \nu}_{\sigma\ \frac{1}{2}\sigma+\frac{1}{3}h_1\ \tau \qquad \rho-i\ \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\ \nu-j} &= (q^{-1}-q)^{j+\frac{1}{2}i} \\
&\times \left\{ \frac{[\frac{1}{3}g_1-\rho+i]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu]![\frac{1}{2}\rho+\frac{1}{3}g_1+\nu]!}{[\frac{1}{3}g_1-\rho]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu+j-\frac{1}{2}i]![\frac{1}{2}\rho+\frac{1}{3}g_1+\nu-j-\frac{1}{2}i]!} \right. \\
&\times \left. \frac{[\frac{1}{3}h_1-\sigma+i]![\frac{1}{2}\sigma+\frac{1}{3}h_1-\tau+j-\frac{1}{2}i]![\frac{1}{2}\sigma+\frac{1}{3}h_1+\tau]!}{[j+\frac{1}{2}i]![\frac{1}{3}h_1-\sigma]![\frac{1}{2}\sigma+\frac{1}{3}h_1-\tau]![\frac{1}{2}\sigma+\frac{1}{3}h_1+\tau-j-\frac{1}{2}i]!} \right\}^{\frac{1}{2}} \\
&\times q^{-\frac{1}{6}g_1 i+\frac{1}{2}i\rho+\frac{1}{2}\nu i-\sigma\rho+\frac{1}{3}g_1\sigma-\sigma\nu-\frac{1}{2}ij+\frac{1}{3}g_1 j+\frac{1}{2}\rho j+\frac{1}{4}i\sigma-\frac{1}{3}h_1 i+\frac{1}{3}h_1\rho+\frac{2}{9}h_1 g_1} \\
&\times q^{-\frac{2}{3}h_1\nu-\frac{2}{3}g_1\tau-\rho\tau+\tau i+\tau j+\frac{1}{4}i+\frac{1}{2}j-\frac{1}{2}j^2-\frac{1}{8}+\frac{3}{4}\rho+\frac{1}{2}g_1-\frac{3}{2}\nu} \\
&\times \sum_z \frac{(-)^z q^{-\frac{2}{3}zh_1-z\sigma+2z\tau-z+\frac{1}{2}zi-zj}[j+\frac{1}{2}i+z]!}{[z]![i-\frac{1}{2}\rho-\frac{1}{3}g_1+\nu+z]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu-z]!}
\end{aligned} \tag{5.21}
$$

To obtain R-matrices with $h_2 \geq 0$ and independent t we use equations (5.4) and (5.4) as follows

$$
\begin{aligned}
\left(R_q^{h_1 h_2\, g_1 0}\right)^{\sigma-i\ \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2-\frac{1}{2}i\ \tau-j \qquad \rho\ \frac{1}{2}\rho+\frac{1}{3}g_1\ \nu}_{\sigma\ \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2\ \tau \qquad \rho-i\ \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\ \nu-j} &= \left\{ \frac{[\frac{2}{3}h_1-\frac{1}{3}h_2+\sigma-i+1]}{[\frac{2}{3}h_1-\frac{1}{3}h_2+\sigma]} \right\}^{\frac{1}{2}} \\
&\times q^{\rho-\frac{1}{2}i}\left(R_q^{h_1 h_2-1\, g_1 0}\right)^{\sigma+\frac{2}{3}-i\ \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2-\frac{1}{2}i\ \tau-j \qquad \rho\ \frac{1}{2}\rho+\frac{1}{3}g_1\ \nu}_{\sigma+\frac{2}{3}\ \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2\ \tau \qquad \rho-i\ \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\ \nu-j}
\end{aligned} \tag{5.22}
$$

On iterating h_2 times

$$
\begin{aligned}
\left(R_q^{h_1 h_2\, g_1 0}\right)^{\sigma-i\ \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2-\frac{1}{2}i\ \tau-j \qquad \rho\ \frac{1}{2}\rho+\frac{1}{3}g_1\ \nu}_{\sigma\ \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2\ \tau \qquad \rho-i\ \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\ \nu-j} &= \\
q^{h_2\rho-\frac{1}{2}h_2 i} &\left\{ \frac{[\frac{2}{3}h_1+\frac{2}{3}h_2+\sigma-i]![\frac{2}{3}h_1-\frac{1}{3}h_2+\sigma]!}{[\frac{2}{3}h_1-\frac{1}{3}h_2+\sigma-i]![\frac{2}{3}h_1+\frac{2}{3}h_2+\sigma]!} \right\}^{\frac{1}{2}} \\
&\times \left(R_q^{h_1 0\, g_1 0}\right)^{\sigma+\frac{2}{3}h_2-i\ \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2-\frac{1}{2}i\ \tau-j \qquad \rho\ \frac{1}{2}\rho+\frac{1}{3}g_1\ \nu}_{\sigma+\frac{2}{3}h_2\ \frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2\ \tau \qquad \rho-i\ \frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\ \nu-j}
\end{aligned} \tag{5.23}
$$

and substituting for the R-matrix on the right-hand side from equation (5.21) we have

$$\left(R_q^{h_1 h_2\, g_1 0}\right)^{\sigma-i\;\frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2-\frac{1}{2}i\;\tau-j\qquad \rho\;\frac{1}{2}\rho+\frac{1}{3}g_1\;\nu}_{\sigma\;\frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2\;\tau\qquad \rho-i\;\frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\nu-j} = (q^{-1}-q)^{j+\frac{1}{2}i}$$

$$\times\, q^{\frac{1}{3}g_1(-\frac{1}{2}i+\sigma+j-2\tau)+(\frac{1}{3}h_1+\frac{1}{3}h_2)(\frac{2}{3}g_1+\rho-2\nu-i)+\frac{1}{2}\rho i+\frac{1}{2}\nu i-\sigma\rho-\sigma\nu-\frac{1}{2}ij+\frac{1}{2}\rho j+\frac{1}{4}i\sigma-\rho\tau+\tau i+\tau j}$$

$$\times\, q^{\frac{1}{4}i+\frac{1}{2}j-\frac{1}{2}j^2-\frac{1}{8}i^2+\frac{3}{4}\rho+\frac{1}{2}g_1-\frac{3}{2}\nu}\sum_z \frac{(-)^z q^{-\frac{2}{3}zh_1-z\sigma+2z\tau-z+\frac{1}{2}zi-zj}[j+\frac{1}{2}i+z]!}{[z]![i-\frac{1}{2}\rho-\frac{1}{3}g_1+\nu+z]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu-z]!}$$

$$\times\left\{\frac{[\frac{1}{3}g_1-\rho]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu]![\frac{1}{2}\rho+\frac{1}{3}g_1+\nu]!}{[\frac{1}{3}g_1-\rho]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu+j-\frac{1}{2}i]![\frac{1}{2}\rho+\frac{1}{3}g-1+\nu-j-\frac{1}{2}i]!}\right.$$

$$\times\frac{[\frac{1}{3}h_1-\frac{2}{3}h_2-\sigma+i]![\frac{2}{3}h_1-\frac{1}{3}h_2+\sigma]![\frac{2}{3}h_1+\frac{2}{3}h_2+\sigma-i]!}{[j+\frac{1}{2}i]![\frac{1}{3}h_1-\frac{2}{3}h_2-\sigma]![\frac{2}{3}h_1+\frac{2}{3}h_2+\sigma]![\frac{2}{3}h_1-\frac{1}{3}h_2+\sigma-i]!}$$

$$\left.\times\frac{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-\tau+j-\frac{1}{2}i][\frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2+\tau]!}{[\frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2-\tau]![\frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2+\tau-j-\frac{1}{2}i]!}\right\}^{\frac{1}{2}} \tag{5.24}$$

To obtain R-matrices with independent t, we use equation (5.4)

$$\left(R_q^{h_1 h_2\, g_1 0}\right)^{\sigma-i\;t-\frac{1}{2}i\;\tau-j\qquad \rho\;\frac{1}{2}\rho+\frac{1}{3}g_1\;\nu}_{\sigma\;t\;\tau\qquad \rho-i\;\frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\nu-j} = q^{-\rho+\frac{1}{2}i}\left\{\frac{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+1]}{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t-i+1]}\right\}^{\frac{1}{2}}$$

$$\times\left(R_q^{h_1-1\, h_2-1\, g_1 0}\right)^{\sigma-\frac{2}{3}-\frac{1}{2}i\;t-\frac{1}{2}i\;\tau-j\qquad \rho\;\frac{1}{2}\rho+\frac{1}{3}g_1\;\nu}_{\sigma-\frac{2}{3}\;t\;\tau\qquad \rho-i\;\frac{1}{2}\rho-\frac{1}{2}i\;\nu-j} \tag{5.25}$$

and iterating $\frac{1}{2}\sigma+\frac{1}{3}h_1+\frac{1}{3}h_2-t$ times

$$\left(R_q^{h_1 h_2\, g_1 0}\right)^{\sigma-i\;t-\frac{1}{2}i\;\tau-j\qquad \rho\;\frac{1}{2}\rho+\frac{1}{3}g_1\;\nu}_{\sigma\;t\;\tau\qquad \rho-i\;\frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\nu-j} =$$

$$q^{-\frac{1}{2}\sigma\rho-\frac{1}{3}h_1\rho-\frac{1}{3}h_2\rho+t\rho+\frac{1}{4}\sigma i+\frac{1}{6}h_1 i+\frac{1}{6}h_2 i-\frac{1}{2}ti}\left\{\frac{[2t-i+1]![\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+1]!}{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t-i+1]![2t+1]!}\right\}^{\frac{1}{2}}$$

$$\times\left(R_q^{h'_1 h'_2\, g_1 0}\right)^{\sigma'-i\;\frac{1}{2}\sigma'+\frac{1}{3}h'_1+\frac{1}{3}h'_2-\frac{1}{2}i\;\tau-j\qquad \rho\;\frac{1}{2}\rho+\frac{1}{3}g_1\;\nu}_{\sigma'\;\frac{1}{2}\sigma+\frac{1}{3}h'_1+\frac{1}{3}h'_2\;\tau\qquad \rho-i\;\frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\nu-j} \tag{5.26}$$

and substituting for the R-matrix on the right-hand side

$$\left(R_q^{h_1 h_2\, g_1 0}\right)^{\sigma-i\;t-\frac{1}{2}i\;\tau-j\qquad \rho\;\frac{1}{2}\rho+\frac{1}{3}g_1\;\nu}_{\sigma\;t\;\tau\qquad \rho-i\;\frac{1}{2}\rho+\frac{1}{3}g_1-\frac{1}{2}i\;\nu-j} = (q^{-1}-q)^{j+\frac{1}{2}i}$$

$$\times\, q^{-\frac{1}{6}g_1 i+\frac{1}{2}i\rho+\frac{1}{2}i\nu-\frac{3}{2}\sigma\rho+t\rho+\frac{3}{4}\sigma i-ti+\frac{2}{3}g_1 t-2t\nu-\frac{1}{2}ij+\frac{1}{3}g_1 j+\frac{1}{2}\rho j-\frac{2}{3}g_1\tau-\rho\tau+\tau i+\tau j}$$

$$\times\, q^{\frac{1}{4}i+\frac{1}{2}j-\frac{1}{2}j^2-\frac{1}{8}i^2+\frac{3}{4}\rho+\frac{1}{2}g_1-\frac{3}{2}\nu}\sum_z \frac{(-)^z q^{-2zt+2z\tau-z+\frac{1}{2}zi-zj}[j+\frac{1}{2}i+z]!}{[z]![i-\frac{1}{2}\rho-\frac{1}{3}g_1+\nu+z]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu-z]!}$$

$$\times\left\{\frac{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t]![\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+1]![\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+i]!}{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t-i]![\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t-i+1]![\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t]!}\right.$$

$$\times\frac{[\frac{1}{3}g_1-\rho+i]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu]![\frac{1}{2}\rho+\frac{1}{3}g_1+\nu]!}{[\frac{1}{3}g_1-\rho]![\frac{1}{2}\rho+\frac{1}{3}g_1-\nu+j-\frac{1}{2}i]![\frac{1}{2}\rho+\frac{1}{3}g_1+\nu-j-\frac{1}{2}i]!}$$

$$\left.\times\frac{[t-\tau+j-\frac{1}{2}i]![t+\tau]![2t-i]![2t-i+1]!}{[t-\tau]![t+\tau-j-\frac{1}{2}i]![2t]![2t+1]![j+\frac{1}{2}i]!}\right\}^{\frac{1}{2}} \tag{5.27}$$

The previous two steps can be repeated using the crossing symmetry of the R-matrix, equation (3.28), to obtain the R-matrix with $g_2 \geq 0$ and independent s

$$
\begin{aligned}
&\left(R_q^{h_1\, h_2\, g_1\, g_2}\right)^{\sigma-i\; t-\frac{1}{2}i\; \tau-j \qquad \rho\; s\; \nu}_{\qquad \sigma\; t\; \tau \qquad \rho-i\; s-\frac{1}{2}i\; \nu-j} \\
&= q^{-\frac{3}{2}\sigma\rho+\frac{3}{4}i\sigma+\frac{3}{4}i\rho-\frac{1}{2}ij+\frac{1}{4}i+\frac{1}{2}j-\frac{1}{8}i^2-\frac{1}{2}j^2+2ts-2t\nu-2s\tau-\frac{1}{2}si+\frac{1}{2}\nu i-it+i\tau+j\tau+js+s-\nu} \\
&\quad \times (q^{-1}-q)^{j+\frac{1}{2}i} \sum_z \frac{q^{-2tz+2\tau z+\frac{1}{2}iz-jz-z}(-)^{z-s+\nu}[j+\frac{1}{2}i+z]!}{[z]![s-\nu-z]![-s+\nu+i+z]![j+\frac{1}{2}i]!} \\
&\quad \times \left\{ \frac{[2t-i]![2t-i+1]![t+\tau]![t-\frac{1}{2}i-\tau+j]![\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t]!}{[2t]![2t+1]![t-\tau]![t-\frac{1}{2}i+\tau-j]![\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t-i]!} \right. \\
&\qquad \times \frac{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+1]![\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+i]!}{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t-i+1]![\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t]!} \\
&\qquad \times \frac{[2s-i]![2s-i+1]![s+\nu]![s-\nu]![\frac{2}{3}g_1-\frac{1}{3}g_2-\frac{1}{2}\rho-s+i]!}{[2s]![2s+1]![s-\frac{1}{2}i+\nu-j]![s-\frac{1}{2}i-\nu+j]![\frac{2}{3}g_1-\frac{1}{3}g_2-\frac{1}{2}\rho-s]!} \\
&\qquad\qquad \left. \times \frac{[\frac{1}{3}g_1-\frac{2}{3}g_2+\frac{1}{2}\rho+s]![\frac{1}{3}g_1+\frac{1}{3}g_2+\frac{1}{2}\rho+s+1]!}{[\frac{1}{3}g_1-\frac{2}{3}g_2+\frac{1}{2}\rho+s-i]![\frac{1}{3}g_1+\frac{1}{3}g_2+\frac{1}{2}\rho+s-i+1]!} \right\}^{\frac{1}{2}}
\end{aligned}
\tag{5.28}
$$

the only restrictions are now $l = m = \frac{1}{2}i$ and $j \geq 0$.

The expression above gives a large class of R-matrices including all of those R-matrices found by Ma (1990a, 1990b). For $i = 0$, it factorizes into the $su(3)_q$ part $q^{-\frac{3}{2}\sigma\rho}$ and the $su(2)_q$ R-matrix given in equation (4.44) with $t \to j_1$, $\tau \to m_1$, $j \to i$ etc.

Chapter 6

Knots, braids and knot polynomials

A knot is a single closed loop of string or in more mathematical terms, an embedding of the circle $\mathbb{S}^1$ in the sphere $\mathbb{S}^3$. A link consists of two or more intertwined knots. Tait (1898) was one of the first to study knots in detail, thinking they may have a relation to classifying atomic structure. Although he was wrong in this assumption, physicists still have an interest in knot theory. In more recent times they have looked at knotted strings and applied knot theory to quantum field theory and vice versa. Knots are also finding application in the study of DNA (see Kauffman, 1991, for a review).

Mathematicians have been primarily interested in classifying knots and looking at topological properties of knots. Tait's table of knots (Tait, 1898) contains many duplications. Rolfsen's table (Rolfsen, 1976) still contains one duplication (10_{163} is equivalent to 10_{162}) but we will use this classification in the following sections.

In order to distinguish and classify knots it is useful to find a knot invariant, a function which is unique to a particular knot. The classification used by Rolfsen is by the number of crossings, with knots having the same number of crossings then being further numbered. The number of knots with the same number of crossings grows rapidly as the crossing number increases.

Alexander was the first to associate a polynomial with knots (Alexander, 1928). The polynomial depends only on the topological properties of the knot rather than on a particular diagram of the knot. However, two non-equivalent knots may have the same Alexander polynomial, indeed there are infinitely many knots having the same polynomial as the trivial knot (the circle or unknot). The Alexander polynomial as it was originally defined was difficult to calculate. But in 1970, Conway solved this problem by proving a certain relation, the skein relation, for the Alexander polynomial and thus making it possible to recursively calculate Alexander polynomials (Conway, 1970).

The next major advance was in 1985 when Jones found a new polynomial for knots (Jones, 1985). Braids had been used for describing knots but Jones found a new matrix representation for braids on which a trace could be defined to give a knot polynomial. This polynomial, unlike the Alexander polynomial, is sensitive to taking the mirror image of a knot. Knots which are equivalent to their mirror image are called amphichiral, the remainder being chiral. The Jones polynomial does predict amphichirality but not always correctly and some topologically distinct knots have the same Jones polynomial.

Since 1985, further braid representations have been found with appropriate traces. This has led to many new polynomials, beginning with the two-variable HOMFLY polynomial of which the Jones and Alexander polynomials are special cases (Freyd *et al*, 1985; Kauffman, 1990).

In this chapter, we review the properties of knots and braids. The process of describing a knot by a braid and then of finding a knot polynomial based on a braid representation is outlined.

6.1 Knot diagrams

A knot can be represented by a two-dimensional diagram with finitely many points where two strings cross. At these points, the overcrossing and undercrossing strings need to be distinguished. We use the convention that the overcrossing string is shown as continuous over the crossing point while the undercrossing string is discontinuous, for example, . Any diagram of this form generates a unique knot. The converse is not true.

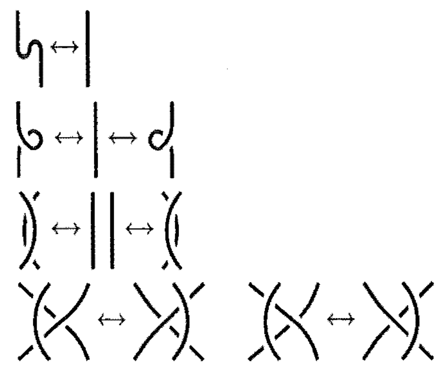
Two knots are said to be ambient isotropic or equivalent if their diagrams can be deformed into another by the following Reidemeister moves

(0)

(I)

(II)

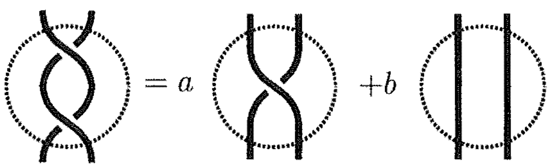
(III)



To illustrate a knot diagram and the Reidmeister moves, we show in Figure 6.1 that the knot 6_3 is amphichiral by showing its equivalence to its mirror image.

6.2 Knot polynomials

A knot polynomial associates with each knot diagram a polynomial which is invariant under the Reidmeister moves. Thus the polynomial describes the knot not just the diagram. Polynomials may be defined by means of a trace or determinant of a matrix describing a knot or by a skein relation or both. A skein relation gives the polynomial of a knot in terms of a sum of polynomials for knots differing only in a certain part of their diagrams. For example, the skein relations for the Alexander, Jones and HOMFLY polynomials are of the form below where the knot diagrams are the same except in the dotted circle.



(6.1)

The coefficients a, b depend on the polynomial being considered.

The knot diagrams on the right-hand side are simpler than the original. In the case of a two-term skein relation a polynomial for a knot can always be obtained from that of the unknot by successively reducing the number of crossings.

6.3 Braids

Braids provide a means of mathematically describing knots. An m-braid is a set of m strings between two parallel sets of m points arranged horizontally, as illustrated in Figure 2(a). Braids were first discussed in terms of the braid group by Artin (1925). The set of m-braids form a group, B_m, with concatenation as the group operation. An m-braid is generated by the set of single twists b_i and b_i^{-1} for $1 \leq i \leq m-1$ as shown in Figures 2(b) and 2(c).

An equivalent description of the braid group B_m (Artin, 1925) is that its generators satisfy the following relations, illustrated in Figure 6.3,

$$b_i b_j = b_j b_i, \quad |i-j| \geq 2 \tag{6.2}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}. \tag{6.3}$$

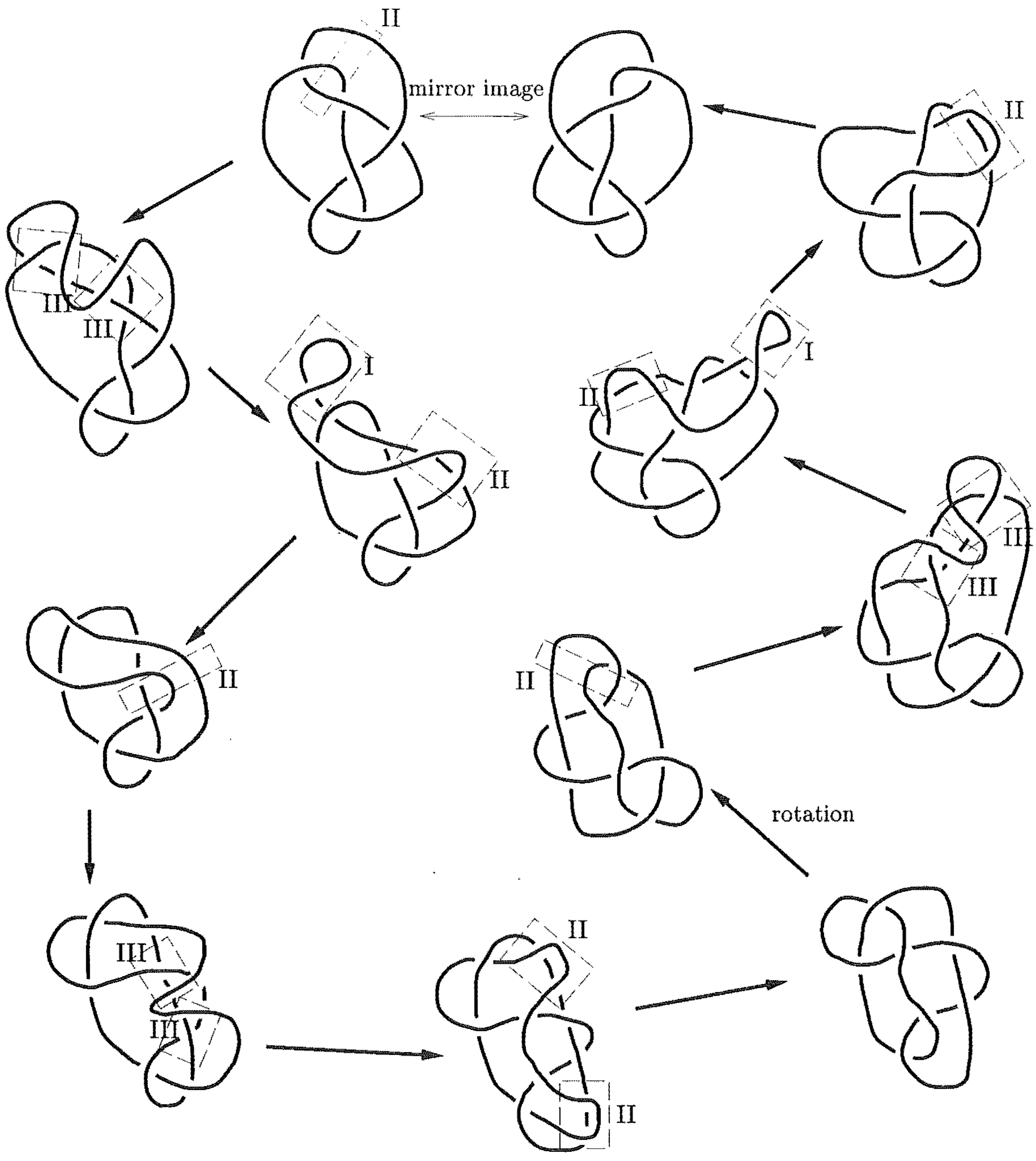
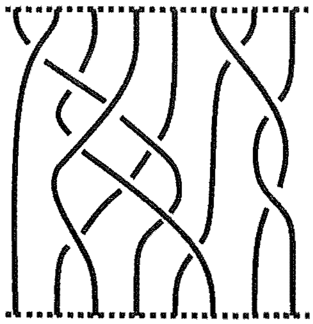
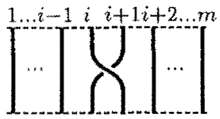


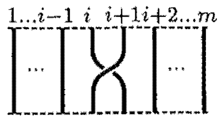
Figure 6.1: Equivalence of the knot 6_3 and its mirror image with Reidmeister moves marked



(a) An 8-braid

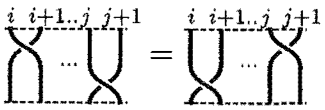


(b) b_i

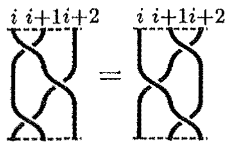


(c) b_i^{-1}

Figure 6.2: Braids and their generators



(a) $b_i b_j = b_j b_i$



(b) $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$

Figure 6.3: Braid group relations

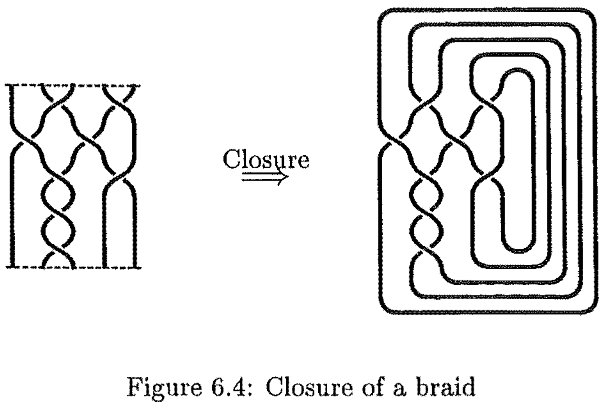
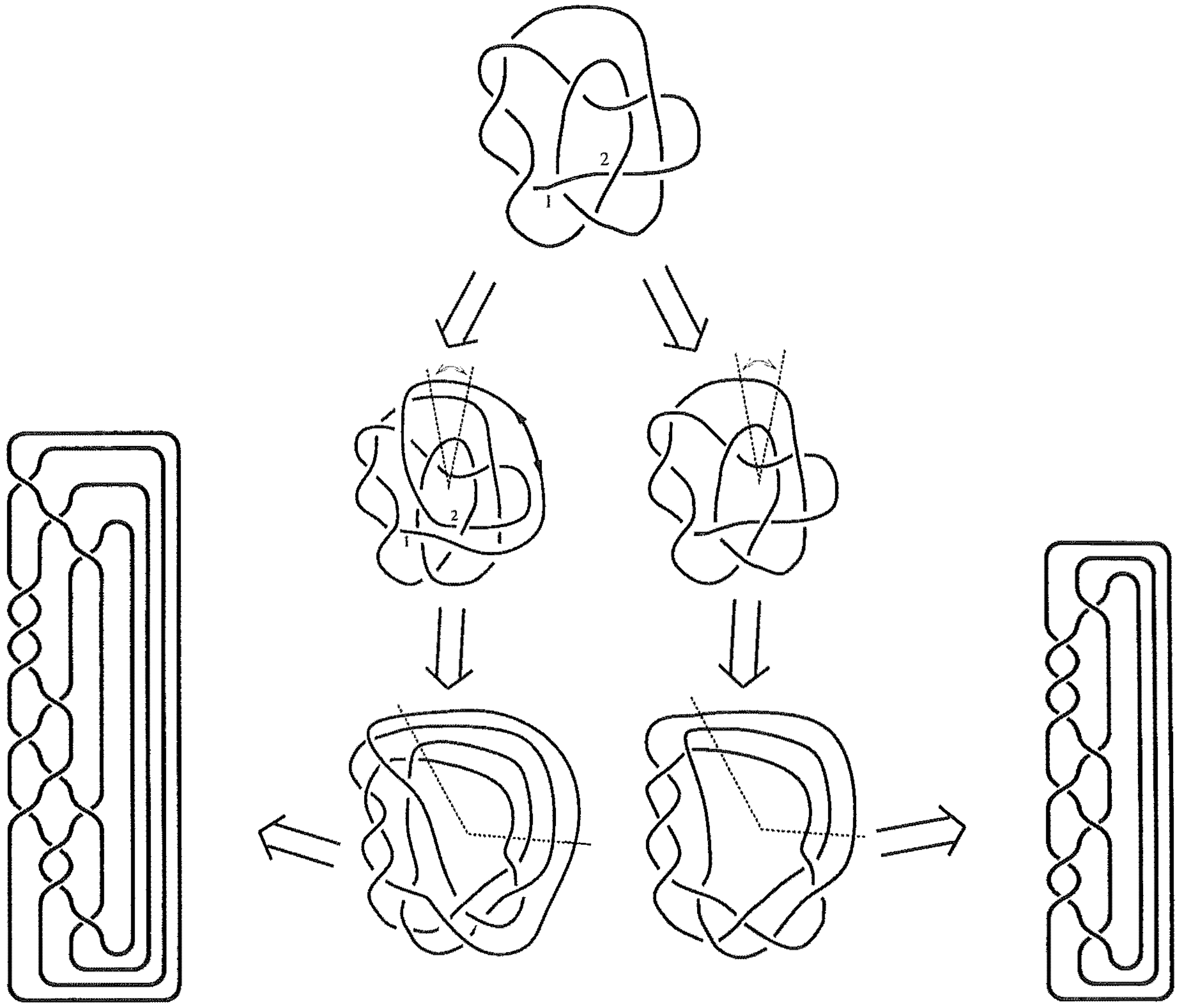


Figure 6.4: Closure of a braid



(a) $b_1 b_2 b_3 b_1^{-3} b_2 b_1^{-1} b_2^{-1} b_3 b_1^{-1} b_2^{-2} b_3$ (b) $b_2 b_1^{-3} b_2 b_1^{-1} b_2 b_1^{-2} b_2$

Figure 6.5: Two braid representations of the knot 10_{106}, differing due to the section between the points marked 1 and 2 being looped across in (a)

Given a braid, its closure is found by connecting the points at top and bottom, i to i, as shown in Figure 6.4. It is clear that the closure of a braid gives a knot or link. Alexander (1923) proves that any knot or link can be described by the closure of a braid. The description of knots by closed braids is highly non-unique, as illustrated in Figure 6.5. Conjugate braids, those which are related by the braid group relations, equations (6.2) and (6.3), give rise to equivalent knots. Different conjugacy classes of braids may still give the same knot on closure. The minimum dimension of braid needed to describe a knot is called the braid index of a knot. Even such a minimum conjucacy class of braid is non-unique. There is also no general relation for the braid index of a knot although bounding values can be obtained.

However, Markov (1945) establishes that if two braids A' and A'' are obtained from the braid $A \in B_m$ by the moves given below, then the knots formed by the closure of the braids A, A', and A'' are topologically equivalent

$$\begin{aligned} \text{Markov I} \quad A' &= BAB^{-1}, \text{ for an arbitrary braid } B \in B_m && (6.4) \\ \text{Markov II} \quad A'' &= Ab_m^{\pm 1}, \text{ for } b_m, b_m^{-1} \in B_{m+1} && (6.5) \end{aligned}$$

These properties enable a knot polynomial to be found from the braid group description of a knot. A knot polynomial $\alpha(A)$ is independent of the braid group description used, depending only on the knot, if it satisfies $\alpha(A) = \alpha(A') = \alpha(A'')$, for A', A'' as given in equations (6.4) and (6.5) and satisfies $\alpha(A) = \alpha(\overline{A})$ for braids $\overline{A}$ of the same conjucacy class as A.

If a function ϕ, called a Markov trace, on matrix representations A, B of braids A, B can be defined so that it satisfies

$$\phi(\mathsf{A}) = \phi(\mathsf{BAB}^{-1}) \tag{6.6}$$

$$\phi(\mathsf{Ab}_m) = \tau\phi(\mathsf{A}), \quad \phi(\mathsf{Ab}_m^{-1}) = \overline{\tau}\phi(\mathsf{A}) \quad \text{for } A \in B_m,\ b_m, b_m^{-1} \in B_{m+1} \tag{6.7}$$

for τ and $\overline{\tau}$ such that $\tau = \phi(\mathsf{b}_i)$ and $\overline{\tau} = \phi(\mathsf{b}_i^{-1})$ for all i, then a knot polynomial can easily be obtained. The knot polynomial for the knot formed by the closure of the braid A is then

$$\alpha(A) = (\tau\overline{\tau})^{-(m-1)/2} \left(\frac{\overline{\tau}}{\tau}\right)^{e(A)/2} \phi(\mathsf{A}) \tag{6.8}$$

where $A \in B_m$ and $e(A)$ is the sum of the exponents of the generators b_i in A (Akutsu and Wadati, 1987a). It is easily shown to be invariant under the Markov moves, equations (6.4) and (6.5).

There may be a further relation between the braid group representations of the form

$$\mathsf{b}_i^k = g_{k-1}\mathsf{b}_i^{k-1} + \cdots + g_1\mathsf{b}_i + g_0. \tag{6.9}$$

This leads to the generalized Alexander-Conway skein relation for a knot polynomial based on this representation for the braid group

$$\alpha(\mathsf{Ab}_i^k\mathsf{B}) = h_{k-1}\alpha(\mathsf{Ab}_i^{k-1}\mathsf{B}) + \cdots + h_1\alpha(\mathsf{Ab}_i\mathsf{B}) + h_0\alpha(\mathsf{AB}) \text{ for arbitrary braids } A,\ B \tag{6.10}$$

For $k = 2$, this is the skein relation shown in equation (6.1).

The degree, k, of the skein relation is called the power of the polynomial and is denoted N. The Jones polynomial and its two-variable extensions, for example, have $N = 2$. Skein relations are used in the following chapter as recursion relations.

In this thesis, we make use of other skein-type relations of form

$$\alpha(\mathsf{AC}) = l_{k-1}\alpha(\mathsf{Ab}_m^k) + \ldots + l_1\alpha(\mathsf{Ab}_m) + l_0\alpha(\mathsf{A})$$
$$\text{for arbitrary } A \in B_m, \text{ and constant braid } C \tag{6.11}$$

These relations are less powerful than the Alexander-Conway skein relation but together with the skein relation enable higher power polynomials to be calculated.

Chapter 7

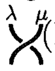
Knot polynomials based on representations of $su(n)_q$

R-matrices give representations of braid groups. Furthermore their other special properties allow a Markov trace and hence knot polynomials to be obtained. In this chapter, we describe recursion techniques for finding knot polynomials based on R-matrices of $su(n)_q$. We calculate the $\{1\}$ $su(n)_q$ polynomials using the skein relation as a recursion relation. Because the $\{1\}$ $su(n)_q$ skein relation is of power $N = 2$ this is sufficient to obtain all knot polynomials.

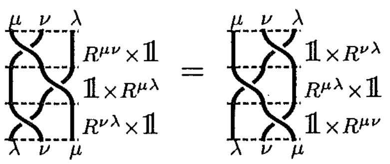
For polynomials of power $N > 2$, the skein relation is insufficient as a recursion relation. However, other relations can be obtained by manipulating the product of R-matrices describing the braid representation of a knot. The relations described in Chapter 3 for coupling coefficients and R-matrices are used to simplify the product. This process is carried out in diagrammatic form. We use this technique to calculate polynomials based on the $\{2\}$ representation of $su(n)_q$. Only a few $\{2\}$ $su(2)_q$ polynomials were previously known (Akutsu *et al*, 1987). Calculating a larger class allows a more accurate evaluation of the $N = 3$ polynomial.

7.1 Braid representations and Markov traces

The R-matrix $R^{\lambda\mu}$ gives a twisting on the two spaces. It can be given in diagrammatic form as 

with $(R^{\lambda\mu})^{-1}$ being 

(Reshetikhin, 1987; Hou *et al*, 1990a). The Yang-Baxter equation, (3.31), can be given diagrammatically as



(7.1)

where the diagram is read top to bottom.

Akutsu and Wadati (1987a) recognized that the Yang-Baxter equation is equivalent to the braid group equation (6.3): compare equation (7.1) with Figure 3(b). The general R-matrices $R^{\lambda\mu}$ give representations of a more general braid group than that described in section 6.3. Each string is associated with a weight or colour, in this case a representation of $su(n)_q$. However, in order for closure to be meaningful we require that the colour in the i^{th} position must be the same top and bottom. Hence for braids which on closure give a knot, all colours are the same and the braid depends on a single representation. This is the situation of section 6.3.

A matrix representation, b_i, for a generator b_i of the braid group B_m can be defined by

$$\mathsf{b}_i = \underbrace{\mathbb{1} \times \cdots \times \overbrace{R^{\lambda\lambda}}^{i\ i+1} \times \cdots \times \mathbb{1}}_{m} \tag{7.2}$$

with equation (6.2) being satisfied trivially and equation (6.3) following from the Yang-Baxter equation (3.31). The matrices $\mathbb{1}$ are of order equal to the dimension of the weight space of λ.

A Markov trace can be obtained for the braid group representation obtained from a representation λ of the q-deformed algebra $su(n)_q$ in the manner just defined (Reshetikhin, 1987). We first define an enhancement matrix V by

$$\mathsf{V} = \underbrace{v_\lambda \times \cdots \times v_\lambda}_{m}, \quad v_\lambda = \operatorname{diag}\{q^{2\rho(\Lambda)} | \Lambda \text{ a weight of } \lambda\} \tag{7.3}$$

where the braid group under consideration has dimension m.

The R-matrix $R^{\lambda\lambda}$ and the matrix v_λ satisfy the following relations

$$\begin{aligned} R^{\lambda\lambda}(v_\lambda \times v_\lambda) &= (v_\lambda \times v_\lambda) R^{\lambda\lambda} && (7.4) \\ \operatorname{tr}_2((\mathbb{1} \times v_\lambda) R^{\lambda\lambda}) &= q^{-c(\lambda)} \mathbb{1} && (7.5) \\ \operatorname{tr}(v_\lambda) &= |\lambda| && (7.6) \end{aligned}$$

where the trace in equation (7.5) is over the second space.

Equation (7.6) is exactly the q-dimension $|\lambda|$ of the representation λ given in equation (3.1). Proofs for equations (7.4) and (7.5) are given in Reshetikhin (1987) and Zhang *et al* (1991). Equation (7.4) follows from the property of the universal R-matrix given by equation (2.4). Equation (7.5) is proved by induction. Suppose it is true for all representations of power k, i.e. representations μ such that $\mu \in \overset{k}{\otimes}\, \epsilon$ then for a representation λ of power $k+1$ we have

$$
\begin{aligned}
&\sum_{i_2} q^{2\rho(i_2)} \left(R_q^{\lambda\,\lambda}\right)_{i_2\ i}^{i_1\ i_2} \\
&= \sum_{i_2 j_1 j_2} q^{2\rho(i_2)}\ {}_q\langle \mu j_1 \epsilon j_2 | \lambda i_1\rangle\ {}_q\langle \lambda i_1 | \mu j_1 \epsilon j_2\rangle \left(R_q^{\lambda\,\lambda}\right)_{i_2\ i}^{i_1\ i_2} \\
&= \sum_{i_2 j_1 j_2 j k_1 k_2} q^{2\rho(i_2)}\ {}_q\langle \mu j_1 \epsilon j_2 | \lambda i_1\rangle \left(R_q^{\epsilon\,\lambda}\right)_{k_2\ j}^{j_2\ i_2} \left(R_q^{\mu\,\lambda}\right)_{k_1\ i}^{j_1\ j}\ {}_q\langle \lambda i_2 | \mu k_1 \epsilon k_2\rangle \\
&= \sum_{j_1 j_2 j k_1 k_2 k l_1 l_2} q^{2\rho(k_1)+2\rho(k_2)}\ {}_q\langle \mu j_1 \epsilon j_2 | \lambda i_1\rangle \left(R_q^{\epsilon\,\mu}\right)_{k\ l_1}^{j_2\ k_1} \left(R_q^{\epsilon\,\epsilon}\right)_{k_2\ l_2}^{k\ k_2}\ {}_q\langle \lambda j | \mu l_1 \epsilon l_2\rangle \left(R_q^{\mu\,\lambda}\right)_{k_1\ i}^{j_1\ j} \\
&= q^{-c(\epsilon)} \sum_{j_1 j_2 k_1 l_1 l_2 l n_1 n_2} q^{2\rho(k_1)}\ {}_q\langle \mu j_1 \epsilon j_2 | \lambda i_1\rangle \left(R_q^{\epsilon\,\mu}\right)_{l_2\ l_1}^{j_2\ k_1} \left(R_q^{\mu\,\mu}\right)_{l\ n_1}^{j_1\ l_1} \left(R_q^{\mu\,\epsilon}\right)_{k_1\ n_2}^{l\ l_2}\ {}_q\langle \lambda i | \mu n_1 \epsilon n_2\rangle \\
&= q^{-c(\epsilon)} \sum_{j_1 j_2 k_1 n_1 n_2 m_1 m_2 m} q^{2\rho(k_1)}\ {}_q\langle \mu j_1 \epsilon j_2 | \lambda i_1\rangle \left(R_q^{\mu\,\epsilon}\right)_{m_1\ m_2}^{j_1\ j_2} \left(R_q^{\mu\,\mu}\right)_{k_1\ m}^{m_1\ k_1} \left(R_q^{\epsilon\,\mu}\right)_{n_2\ n_1}^{m_2\ m}\ {}_q\langle \lambda i | \mu n_1 \epsilon n_2\rangle \\
&= q^{-c(\epsilon)-c(\mu)} \sum_{j_1 j_2 n_1 n_2 m_1 m_2} {}_q\langle \mu j_1 \epsilon j_2 | \lambda i_1\rangle \left(R_q^{\mu\,\epsilon}\right)_{m_1\ m_2}^{j_1\ j_2} \left(R_q^{\epsilon\,\mu}\right)_{n_2\ n_1}^{m_2\ m_1}\ {}_q\langle \lambda i | \mu n_1 \epsilon n_2\rangle \\
&= q^{c(\lambda)} \sum_{j_1 j_2} {}_q\langle \mu j_1 \epsilon j_2 | \lambda i_1\rangle\ {}_q\langle \lambda i | \mu j_1 \epsilon j_2\rangle = q^{c(\lambda)} \delta_{i_1 i}
\end{aligned}
\tag{7.7}
$$

where we have used vector coupling coefficient orthogonality, equation (3.7), the pentagonal equation, equation (3.30), the induction hypothesis, the Yang-Baxter equation, equation (3.31) and equation (7.14). The result follows by induction, as it is true for the primitive representation (Reshetikhin, 1987).

The modified trace defined below is a Markov trace

$$
\phi(A) = \frac{1}{|\lambda|^m} \mathrm{tr}(\mathsf{V}A) \quad \tau = \frac{q^{-c(\lambda)}}{|\lambda|},\ \overline{\tau} = \frac{q^{c(\lambda)}}{|\lambda|} \tag{7.8}
$$

with equation (6.6) following from equation (7.4) and equation (6.7) following from equation (7.5). The knot polynomial for the braid $A \in B_n$ from equations (6.8) and (7.8) is thus

$$
\alpha(A) = |\lambda|^{-1}\, q^{e(A)c(\lambda)} \mathrm{tr}(\mathsf{V}A) \tag{7.9}
$$

The polynomial is normalized so that for the unknot $\alpha(b_i) = \alpha(1) = \alpha(b_i^{-1}) = 1$ where 1 is the trivial (and only) braid in B_1, a single string. Note that $\alpha(b_i^0) = |\lambda|$.

From equation (3.27), the R-matrices $R^{\lambda\lambda}$ have eigenvalues $q^{2c(\lambda)-c(\mu)}$, where $\mu \subset \lambda \times \lambda$, the eigenfunctions being the coupling coefficients. Thus $R_q^{\lambda\lambda}$ satisfies

$$\prod_{\mu\subset\lambda\times\lambda} \left((R^{\lambda\lambda})^r - \{\lambda\lambda\mu^*\}^r \, q^{r(2c(\lambda)-c(\mu))} \right) = 0 \tag{7.10}$$

where r is the multiplicity of μ in $\lambda \times \lambda$. This gives an equation of the form of equation (6.9) for the braid group representations and hence a skein relation for the knot polynomials (Broda, 1991). The power of the polynomial based on a particular representation λ thus depends on the number of terms in the Kronecker product $\lambda \times \lambda$.

7.2 Skein-type relations

In principle, polynomials for knots based on any representation of $su(n)_q$ can be obtained from equation (7.9). However, for all but the simplest knots, the matrix multiplications are long and tedious. The skein relation is sufficient to calculate all polynomials for the fundamental representation of $su(n)_q$ by recursion, since these have power $N = 2$. For other representations, the skein relation is not sufficient for complete calculation but can be used in simplification.

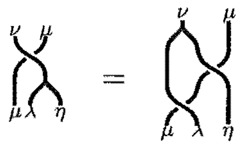
Guadagnini (1992) outlines a new method for calculating $su(n)_q$ knot polynomials. The knot polynomial for a particular representation is reduced to a sum over polynomials for trivial knots based on other representations. The properties of Wilson line operators are used in the simplification process.

In this section, we use a similar approach to Guadagnini but with two major differences. Firstly, rather than tackling each knot individually, we derive two 'skein-type' relations which can be used to calculate polynomials for whole classes of braids. Secondly, the properties of R-matrices and vector coupling coefficients are used to simplify the braid. These arise naturally in the q-deformed algebra.

In a similar manner to the R-matrices, the coupling coefficient $\langle\mu\nu|\lambda\rangle$ and its conjugate $\langle\lambda|\mu\nu\rangle$ are given diagrammatically as  and 

We restate some of the equations of chapter 3, equations (3.30), (3.7) and (3.8) giving their diagrammatic forms also

$$(R^{\nu\mu})^{m_3 m}_{m_3' m''} \langle\lambda m_1' \eta m_2' | \nu m_3'\rangle = \langle\lambda m_1 \eta m_2 | \nu m_3\rangle (R^{\eta\mu})^{m_2 m}_{m_2' m'} (R^{\lambda\mu})^{m_1 m'}_{m_1' m''} \tag{7.11a}$$

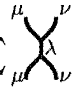


(7.11b)

$$\sum_{i_1 i_2} \langle \mu i_1 \nu i_2 | r \lambda i \rangle \langle r' \lambda' i' | \mu i_1 \nu i_2 \rangle = \delta_{rr'} \delta_{\lambda\lambda'} \delta_{ii'} \tag{7.12a}$$

 = (7.12b)

$$\sum_{r\lambda i} \langle \mu i_1 \nu i_2 | r \lambda i \rangle \langle r \lambda i | \mu i'_1 \nu i'_2 \rangle = \delta_{i_1 i'_1} \delta_{i_2 i'_2} \tag{7.13a}$$

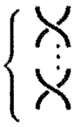
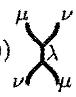
$\sum_{\lambda}$  =  (7.13b)

Equation (3.27) can be generalized using the vector coupling coefficient orthogonality relations, (7.12) and (7.13) to give the following equations

$$\underbrace{(R^{\mu\nu})^{m_1 m_2}_{k_1 k_2} (R^{\nu\mu})^{k_1 k_2}_{k'_2 k'_1} \dots (R^{\nu\mu})^{k''_2 k''_1}_{m'_2 m'_1}}_{2n}$$
$$= \sum_{\lambda \in \mu \times \nu} q^{2n(c(\mu)+c(\nu)-c(\lambda))} \langle \mu m_1 \nu m_2 | \lambda m \rangle \langle \lambda m | \mu m'_1 \nu m'_2 \rangle \tag{7.14a}$$

$2n\Big\{$  $= \sum_{\lambda} q^{2n(c(\mu)+c(\nu)-c(\lambda))}$  (7.14b)

$$\underbrace{(R^{\mu\nu})^{m_1 m_2}_{k_1 k_2} \dots (R^{\mu\nu})^{k''_1 k''_2}_{m'_1 m'_2}}_{2n+1}$$
$$= \sum_{\lambda \in \mu \times \nu} \{\mu\nu\lambda^*\}\, q^{(2n+1)(c(\mu)+c(\nu)-c(\lambda))} \langle \mu m_1 \nu m_2 | \lambda m \rangle \langle \lambda m | \nu m'_1 \mu m'_2 \rangle, \tag{7.15a}$$

$2n+1\Big\{$  $= \sum_{\lambda} \{\mu\nu\lambda^*\}\, q^{(2n+1)(c(\mu)+c(\nu)-c(\lambda))}$  (7.15b)

In order to use the above relations in the simplification process, the more general idea of a braid is used, with strings being associated with different representations. For braids with sensible closure, the Markov trace ϕ generalizes. If the strings are associated with representations μ, ν etc. then $\phi_{\mu\nu\dots} = (|\mu|\,|\nu|\,..)^{-1}\mathrm{tr}(\mathsf{V}_{\mu\nu\dots}\mathsf{A})$ where $\mathsf{V}_{\mu\nu\dots} = v_\mu \times v_\nu \times \dots$.

From the definition of the Markov trace based on $su(n)_q$, taking the closure of a braid is equivalent to taking a modified trace of the product of R-matrices representing the braid. The modified trace can be performed on other matrices, extending the idea of closure. When it is applied to coupling coefficients, the following relation holds

$$\mathrm{tr}(\mathsf{V}_\nu \langle \mu\nu|\lambda\rangle\langle\lambda|\mu'\nu\rangle) = \delta_{\mu\mu'} \frac{|\lambda|}{|\mu|} \tag{7.16a}$$

 $= \frac{|\lambda|}{|\mu|}$  (7.16b)

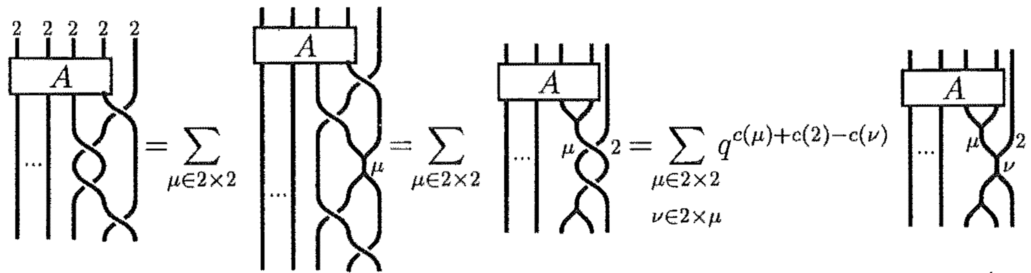
To prove equation (7.16) we use the definition of the enhancement matrix V_μ and the symmetry of the coupling coefficients, equation (3.14), for each of the coupling coefficients and then coupling coefficient orthogonality, equation (3.8)

$$\begin{aligned}\sum q^{2\rho(\mu)}\ {}_q\langle\mu\nu|\lambda\rangle q^{2\rho(\mu)}\ {}_q\langle\lambda|\mu'\nu\rangle &= \frac{|\lambda|}{|\mu|^{\frac{1}{2}}|\mu'|^{\frac{1}{2}}}\sum\ {}_q\langle\mu|\lambda\nu^*\rangle\ {}_q\langle\lambda\nu^*|\mu'\rangle \\ &= \delta_{\mu\mu'}\frac{|\lambda|}{|\mu|}\end{aligned} \tag{7.17}$$

The two skein-type relations used in the calculation of $\{2\}su(n)_q$ polynomials reduce a braid of dimension $m+1$ or $m+2$ to a sum over braids of dimension m. One or two strings may be eliminated by reducing crossings with equations (7.14) and (7.15), manipulating with the pentagonal equation (7.11) and Yang-Baxter equation (7.1) and finally using one of the closure equivalences.

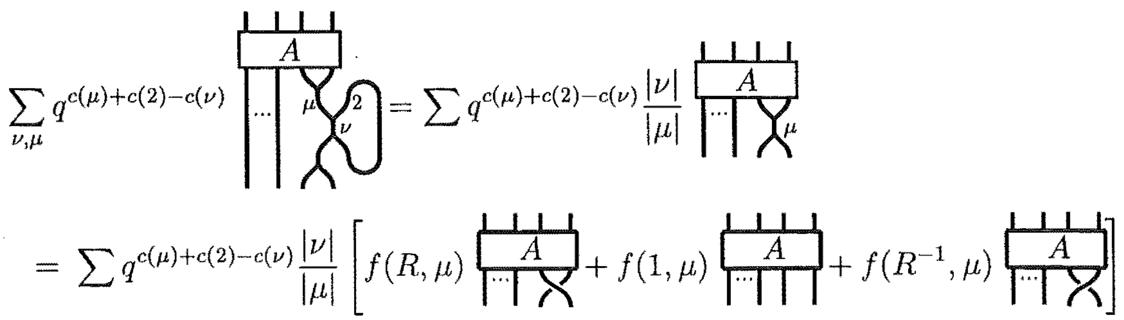
To illustrate this method, the calculation of the skein-type relation, equation (7.23), is outlined below

$\mathsf{A}\mathsf{b}_m\mathsf{b}^2_{m-1}\mathsf{b}_m =$



(7.18)

The first step uses coupling coefficient orthogonality, equation (7.13). The pentagonal equation, equation (7.11) is then repeatedly applied. The final step uses equation (7.14) to write the product of two R-matrices as a second sum over coupling coefficients. On closure and using equation (7.16) we have

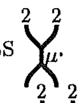
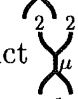
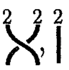
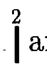
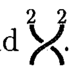


(7.19)

where in the final step the definitions of R^{22} and $(R^{22})^{-1}$, equation (3.27), and the coupling coefficient orthogonality, equation (7.13), are used to give three expressions

Table 7.1: q-dimensions and Casimirs for some representations of $su(n)_q$

Representation	q-dimension	Casimir
$\{1\}$	$[n]$	$\frac{n^2-1}{2n}$
$\{2\}$	$\frac{[n][n+1]}{[2]}$	$\frac{n^2+n-2}{n}$
$\{11\}$	$\frac{[n-1][n]}{[2]}$	$\frac{n^2-n-2}{n}$
$\{4\}$	$\frac{[n][n+1][n+2][n+3]}{[2][3][4]}$	$\frac{2n^2+6n-8}{n}$
$\{31\}$	$\frac{[n-1][n][n+1][n+2]}{[4][2]}$	$\frac{2n^2+2n-8}{n}$
$\{22\}$	$\frac{[n-1][n]^2[n+1]}{[3][2]^2}$	$\frac{2n^2-8}{n}$
$\{6\}$	$\frac{[n][n+1][n+2][n+3][n+4][n+5]}{[2][3][4][5][6]}$	$\frac{3n^2+15n-18}{n}$
$\{51\}$	$\frac{[n-1][n][n+1][n+2][n+3][n+4]}{[2][3][4][6]}$	$\frac{3n^2+9n-18}{n}$
$\{42\}$	$\frac{[n-1][n]^2[n+1][n+2][n+3]}{[2]^2[4][5]}$	$\frac{3n^2+5n-18}{n}$
$\{411\}$	$\frac{[n-2][n-1][n][n+1][n+2][n+3]}{[2]^2[3][6]}$	$\frac{3n^2+3n-18}{n}$
$\{33\}$	$\frac{[n-1][n]^2[n+1]^2[n+2]}{[2]^2[3]^2[4]}$	$\frac{3n^2+3n-18}{n}$
$\{321\}$	$\frac{[n-2][n-1][n]^2[n+1][n+2]}{[3]^2[5]}$	$\frac{3n^2-18}{n}$
$\{222\}$	$\frac{[n-2][n-1]^2[n]^2[n+1]}{[2]^2[3]^2[4]}$	$\frac{3n^2-3n-18}{n}$

for the three products . These equations are rearranged to give the coefficients f for writing the product  in terms of ,  and . Knot polynomials can now be obtained as $\{2\}$ is the only representation involved. The polynomial $\alpha(Ab_m b_{m+1} b_m)$ can thus be written as a sum over the polynomials $\alpha(Ab_m)$, $\alpha(A)$ and $\alpha(Ab_{m-1})$. The q-dimensions and Casimirs for the representations appearing in the products $\mu \in 2 \times 2$, $\nu \in 2 \times \mu$ are given in Table 7.1. The coefficients for both skein-type relations are given in Table 7.2. A similar approach is used for the second skein-type relation, equation (7.24), with two strings being eliminated.

7.3 Knot polynomials based on $\{1\}$ of $su(n)_q$

The fundamental representation of $su(n)_q$ is n-dimensional and can be written in partition form as $\{1, 0, \ldots, 0\}$. In the following sections, it is written as $\{1\}su(n)_q$. The tensor product of the fundamental representation with itself decomposes into two terms, the symmetric term $\{2\}$ and the antisymmetric term $\{1\,1\}$. Using equations (6.8), (7.8) and (7.10), one obtains a two-term skein relation for the $\{1\}su(n)_q$ polynomials as

$$\alpha(Ab_i^2B) = (q^{n-1} - q^{n+1})\alpha(Ab_iB) + q^n\alpha(AB). \tag{7.20}$$

the q-dimensions and Casimirs for the representations appearing are given in Table 7.1.

To calculate the $\{1\}su(n)_q$ polynomial for a knot, the braid representing the knot is written as a product of p terms of form $b_1^{k_1}b_2^{k_2}\dots b_n^{k_n}$, the braid form then appears as $b_1^{l_{11}}b_2^{l_{12}}\dots b_n^{l_{1n}}\dots b_1^{l_{p1}}b_2^{l_{p2}}\dots b_n^{l_{pn}}$. If the last non-zero power is l_{pi}, that is $l_{pi+1} = \dots = l_{pn} = 0$, the polynomial can be written as a sum of polynomials with last non-zero power in the $i-1$ position or lower by means of the skein relation, braid group relations and Markov moves. When $i = 1$, by the Markov move, equation (6.4), the $b_1^{l_{p1}}$ term can be absorbed with the $b_1^{l_{11}}$ term and hence the number of terms p is reduced by 1. The recursion continues until $p = 1$ when $\alpha(b_1^{k_1}\dots b_n^{k_n}) = P_{k_1}P_{k_2}\dots P_{k_n}$, where $P_k = \alpha(b^k)$. For each k, P_k is found from iterating the skein relation with A, B the trivial braids and noting that $P_1 = \alpha(b_1) = P_{-1} = \alpha(b_1^{-1}) = 1$ and that $P_0 = \alpha(b_i^0) = |\lambda|$. The coefficients An and Bn are from the iterated skein relation, $\alpha(Ab_i^nB) = An(n)(Ab_iB) + Bn(n)(AB)$. The full algorithm is given below as Algorithms 7.1, 7.2, 7.3.

Algorithm 7.1 KnotPolynomial

▷ *knot polynomial for the braid $b_1^{k_1}b_2^{k_2}\dots b_n^{k_n}$ can be found directly as $P(k_1)P(k_2)\dots P(k_n)$ where $P(k) = \alpha(b_1^k)$*

▷ *if the braid is not of this form, call the procedure ReduceP to reduce the number of groups*

variables

braid $\leftarrow b_1^{l_{11}}\dots b_1^{l_{21}}\dots b_1^{l_{p1}}\dots b_n^{l_{pn}}$

$p \leftarrow$ number of groups

$n \leftarrow$ braid index $-$ 1

if $n = 1$ **then** $P(l_{11}\cdots + l_{p1})$ ▷ *boundary condition*

elif $p = 1$ **then** $P(l_{11})\dots P(l_{1n})$ ▷ *boundary condition*

else call ReduceP with

braid,n, p, $i \leftarrow n$

end if

Algorithm 7.2 ReduceP

▷ *the procedure looks at the last group, progressively eliminating terms*

▷ *when only the b_1 term is left, it can be combined with the b_1 term of the first group by the Markov I move*

$\triangleright$ *the procedure ReduceI is called to eliminate terms*

variables

braid, p, n

$i \leftarrow$ position of last possible non-zero power i.e. $l_{pi+1} = \ldots = l_{pn} = 0$

if $i = 1$ **then return** to KnotPolynomial with

braid$\leftarrow b_1^{l_{11}+l_{p1}} \ldots b_n^{l_{p-1n}}$

$n, p \leftarrow p-1$ $\triangleright$ *boundary condition*

elif $l_{pi} = 0$ **then return** to ReduceP with

braid $\leftarrow b_1^{l_{11}} \ldots b_{i-1}^{l_{pi-1}}$

$n, p, i \leftarrow i-1$

elif $l_{pi} = 1$ **then call** ReduceI with

braid, n, p, i, $j \leftarrow 1$

else $\triangleright$ $l_{pi} <> 1, 0$

$An(l_{pi})\times$resultA + $Bn(l_{pi})\times$resultB

where

resultA $\leftarrow$ **call** KnotPolynomial with

n, p, braid$\leftarrow b_1^{l_{11}} \ldots b_{i-1}^{l_{pi-1}} b_i$

resultB $\leftarrow$ **call** KnotPolynomial with

n, p, braid $\leftarrow b_1^{l_{11}} \ldots b_{i-1}^{l_{pi-1}}$

end if

Algorithm 7.3 ReduceI

$\triangleright$ *in this procedure, 'shuffles' of the form below are performed*

$$\underbrace{b_k b_{k-1} b_k^l}\, b_{k+1}^{l'} \ldots b_1^m \ldots b_{k-2}^{m'} b_{k-1}^{m''} \ldots$$

$$b_{k-1}^l b_k b_{k-1} b_{k+1}^{l'} \ldots b_1^m \ldots b_{k-2}^{m'} b_{k-1}^{m''} \ldots$$

$\triangleright$ *from successive application of the braid group relation*

$$b_{k-1}^l b_k b_{k+1}^{l'} \ldots b_1^m \ldots b_{k-1} b_{k-2}^{m'} b_{k-1}^{m''} \ldots$$

$\triangleright$ b_{k-1} *commuted through*

$\triangleright$ *if* $m' = 1$ *then the shuffle is repeated with* $k \leftarrow k-1$

$\triangleright$ *if* $m' = 0$ *then the* b_{k-1} *terms can be combined and the shuffling ends*

$\triangleright$ *otherwise the skein relation is used to write the polynomial as a sum of two terms with* $m' = 0$ *and* $m' = 1$

$\triangleright$ *the shuffling ends when* $k = 2$ *(if not before) the two* b_1 *terms are combined*

variables

braid, n, p, i

$j - 1 \leftarrow$ number of 'shuffles'

if $i = j$ **then return** to ReduceP with

braid $\leftarrow b_1^{l_{11}} \ldots b_{i-2}^{l_{1i-2}} b_{i-1}^{l_{1ii}} b_i b_{i+1}^{l_{1i+1}} \ldots b_1^{l_{21}} \ldots b_{i-3}^{l_{2i-3}} b_{i-2}^{l_{2i-1}} b_{i-1} b_i^{l_{2i}} \ldots b_1^{l_{i1}+1} \ldots b_{i-1}^{l_{pi-1}}$

n, p, $i \leftarrow i-1$ ▷ *boundary condition*

elif $l_{j_{\mathrm{mod}_p} i-j} = 0$ **then return** to ReduceP with

braid $\leftarrow b_1^{l_{11}} \dots b_{i-2}^{l_{1i-2}} b_{i-1}^{l_{1ii}} b_i b_{i+1}^{l_{1i+1}} \dots b_1^{l_{21}} \dots$

$\times b_{i-3}^{l_{2i-3}} b_{i-2}^{l_{2i-1}} b_{i-1} b_i^{l_{2i}} \dots b_1^{l_{j_{\mathrm{mod}p} 1}} \dots b_{i-j+1}^{l_{j_{\mathrm{mod}p} i-j+1}+1} \dots b_{i-1}^{l_{pi-1}}$

n, p, $i \leftarrow i-1$ ▷ *boundary condition*

elif $l_{j_{\mathrm{mod}_p} i-j} = 1$ **then return** to ReduceI with

braid, n, p, i, $j \leftarrow j-1$

else ▷ $l_{j_{\mathrm{mod}_p} i-j} <> 1, 0$

$An(l_{j_{\mathrm{mod}_p} i-j}) \times$resultA$+Bn(l_{j_{\mathrm{mod}_p} i-j}) \times$resultB

where

resultA $\leftarrow$ **call** KnotPolynomial with

n, p, braid$\leftarrow b_1^{l_{11}} \dots b_1^{l_{j_{\mathrm{mod}p} 1}} \dots b_{i-j-1}^{l_{j_{\mathrm{mod}p} i-j-1}} b_{i-j} b_{i-j+1}^{l_{j_{\mathrm{mod}p} i-j+1}} \dots b_i^{l_{pi}}$

resultB $\leftarrow$ **call** KnotPolynomial with

n, p, braid $\leftarrow b_1^{l_{11}} \dots b_1^{l_{j_{\mathrm{mod}p} 1}} \dots b_{i-j-1}^{l_{j_{\mathrm{mod}p} i-j-1}} b_{i-j+1}^{l_{j_{\mathrm{mod}p} i-j+1}} \dots b_i^{l_{pi}}$

end if

The knots were sorted according to braid index, i.e. the dimension of the braid representation of the knot, and number of terms p. The initial braid words were taken from Jones (1987) and simplified further via the braid group relations. These simplified braid words are given in Table 7.4. An algebraic package, MAPLE, was then used to carry out the above steps.

For certain values of n, the polynomials have special properties. If $n = 0$, then the polynomials obtained are the Alexander polynomials, as can be seen by comparing the skein relations. For $n = 1$ the polynomials for all knots are equal to 1 and are thus identical. The Jones polynomial corresponds to the $n = 2$ case. For higher values of n, all polynomials are different except for the occasional pair. For $n = 2$, among the 248 knots of ten or fewer crossings, there are 14 pairs of knots having the same polynomials. Of these 14 pairs, 5 are pairs of knots which have the same polynomial for all values of n. For $n > 2$, all knots of ten or fewer crossings were distinguished by the $\{1\}su(n)_q$ polynomial with the exception of these 5 pairs.

The $\{1\}su(n)_q$ polynomials are a special case of the HOMFLY polynomial (Freyd *et al*, 1985). This two-variable polynomial has the skein relation $t^{-1}P(Ab_iB) - tP(Ab_i^{-1}B) = xP(AB)$. With $t = q^n$ and $x = q^{-1} - q$, the skein relation of the $\{1\}su(n)_q$ polynomials is obtained. The pairs of knots of ten or fewer crossings which cannot be distinguished by the $\{1\}su(n)_q$ polynomial are exactly those with the same HOMFLY polynomial. For knots of ten or fewer crossings, the one variable $\{1\}su(n)_q$ polynomials for any $n > 2$ distinguish the same knots as the two variable HOMFLY polynomial.

For special values of the deformation parameter q, the $\{1\}su(n)_q$ polynomials take on certain discrete values. If $q = 1$, i.e. in the non-deformed limit, all the polynomials are equal to 1. Similarly, if q is a certain root of unity, $q^{n-1} = 1$ or $q^{n+1} = -1$, then $\alpha(A) = 1$ for all braids A. This is readily shown from the skein relation. Likewise, using induction on the skein relation, we can show that for $q^{n-1} = -1$ or $q^{n+1} = 1$, the polynomial $\alpha(A)$ equals 1 for braids A that describe a knot or an odd-component link. For the even component case, $\alpha(A)$ equals -1.

These special values mean that for any knot with polynomial $\alpha(A)$, the polynomial $1 - \alpha(A)$ has factors $(q^{n+1} - q^{-(n+1)})$ and $(q^{n-1} - q^{-(n-1)})$. This is used to simplify the knot polynomials. The reduced knot polynomials $\beta(A) = (1 - \alpha(A))/(q^{n+1} - q^{-(n+1)})(q^{n-1} - q^{-(n-1)})$ for knots of ten or fewer crossings are given in Table 7.4. Setting $n = 2$, we recover the table given by Jones (1987) (on replacing q for t).

7.4 Knot polynomials based on $\{2\}$ of $su(n)_q$

Knot polynomials based on the non-fundamental representations of $su(n)_q$ have skein relations of power, N, greater than 2. The Alexander-Conway skein relation is insufficient to determine all knot polynomials. Either the braid representations must be entered explicitly into the definition of the knot polynomial, equation (7.9) or the skein relation must be supplemented with other skein-type relations. All of the $\{2\}su(n)_q$ polynomials for knots of ten or fewer crossings can be calculated using the Alexander-Conway skein relation together with two other similar skein-type relations.

The $\{2\}su(n)_q$ polynomials are of power $N = 3$, as there are three terms in the product

$$\{2\} \times \{2\} = \{4\} + \{31\} + \{22\} \tag{7.21}$$

From equations (6.8), (7.8) and (7.10), the $\{2\}su(n)_q$ skein relation is

$$\alpha(Ab_i^2B) = q^{2n}(q^4 - q^2 + q^{-2})\alpha(Ab_iB) + q^{4n}(q^6 - q^2 + 1)\alpha(AB) - q^{6n+4}\alpha(Ab_i^{-1}B) \tag{7.22}$$

The two relations below are needed in addition to calculate $\{2\}su(n)_q$ polynomials for all knot of ten or fewer crossings where the coefficients $k_1, \ldots\ h_{-1}$ are given in Table 7.2.

$$\alpha(Ab_mb_{m-1}^2b_m) = h_1\alpha(Ab_{m-1}) + h_0\alpha(A) + h_{-1}\alpha(Ab_{m-1}^{-1}), \text{ where } A \in B_m \tag{7.23}$$

Table 7.2: Coefficients for skein-type relations, equations (7.23) and (7.24)

h_1	$-q^{3n+2}+q^{2n+4}-2q^{2n+3}+q^{2n+2}+2q^{2n+1}-2q^{2n}+q^{2n-2}$
h_0	$q^{5n+6}-q^{4n+8}-q^{4n+5}+q^{4n+4}-q^{4n+2}+2q^{3n+8}-3q^{3n+7}-2q^{3n+6}+10q^{3n+5}-6q^{3n+4}-7q^{3n+3}+10q^{3n+2}-q^{3n+1}-4q^{3n}+2q^{3n-1})/(q-1)^2(q+1)$
h_{-1}	$q^{5n+4}-2q^{4n+4}+2q^{4n+3}+q^{4n+2}-2q^{4n+1}$
k_1	$2q^{10n+10}+(-q^{12}-q^{11}-6q^{10}+4q^9+5q^8-5q^7-2q^6)q^{9n}+(4q^{12}-6q^{10}+6q^9-4q^7+8q^6-6q^4+2q^3+2q^2)q^{8n}+(-q^{13}+q^{12}-q^{11}+q^{10}-q^9+2q^7-5q^6+q^5+4q^4-2q^3-q^2-q+1+q^{-1}-q^{-2})q^{7n}/(q-1)^2(q+1)$
k_0	$q^{10n+12}+(-2q^{14}-2q^{11}+2q^{10}-2q^8)q^{9n}+(q^{16}+5q^{14}-6q^{13}-9q^{12}+28q^{11}-7q^{10}-28q^9+23q^8+8q^7-13q^6+2q^5+2q^4)q^{8n}+(-4q^{16}+4q^{15}+10q^{14}-22q^{13}-2q^{12}+30q^{11}-20q^{10}-6q^9+20q^8-24q^7+26q^5-12q^4-12q^3+8q^2+2q-2)q^{7n}+(q^{17}-2q^{16}+4q^{14}-q^{13}-5q^{12}-2q^{11}+17q^{10}-10q^9-19q^8+30q^7-5q^6-20q^5+16q^4-2q^3+q-7+6q^{-1}+q^{-2}-3q^{-3}+q^{-4})q^{6n}/(q-1)^4(q+1)^2$
k_{-1}	$-2q^{8n+8}+(4q^{10}-4q^9+q^8+7q^7-4q^6-2q^5+3q^4+q^3)q^{7n}+(-2q^{12}+2q^{11}+2q^{10}-6q^9+4q^5-2q^4-6q^3+4q-2q^{-1})q^{6n}+(q^{13}-2q^{12}+3q^{10}+q^9-5q^8+q^7+5q^6-5q^5+2q^4+4q^3-5q^2+q+3q^{-2}-2q^{-3}-q^{-4}+q^{-5})q^{5n}/(q-1)^2(q+1)$

$$\alpha(Ab_m b_{m-1} b_{m+1} b_m^2 b_{m-1} b_{m+1} b_m) = k_1\alpha(Ab_{m-1}) + k_0\alpha(A) + k_{-1}\alpha(Ab_{m-1}^{-1}), \quad \text{where } A \in B_m \tag{7.24}$$

The method used in obtaining these relations is outlined in Section 7.2 with equation (7.23) used as an illustration.

The $\{2\}su(n)_q$ knot polynomials are calculated in a similar manner to the $\{1\}su(n)_q$ polynomials. The skein relation, equation (7.22), is used to reduce the knot polynomial to a sum over simpler knot polynomials with the two skein-type relations, equations (7.23) and (7.24), being used when no other simplification is possible.

The $\{2\}su(n)_q$ polynomial can be factored in a similar manner to the $\{1\}su(n)_q$ polynomials. For any braid A, it follows from the skein relation, equation (7.22), and the two skein-type relations, equations (7.23) and (7.24), that $\alpha(A) = 1$ for $q^{2n+4} = 1$ or $q^{2n-2} = 1$. Thus any knot with polynomial $\alpha(A)$ has factors $(q^{(n+2)} - q^{-(n+2)})(q^{(n-1)} - q^{-(n-1)})$ for $1 - \alpha(A)$. This provides a check on the calculation of the polynomials.

Hou *et al* (1990a) show that the $\{2\}su(2)_q$ polynomial is equivalent to the polynomial found from a three-state exactly solvable model in statistical mechanics and given by Akutsu and Wadati (1987b); Akutsu *et al* (1987). The knot polynomials calculated here for knots with braid index 2 or 3 were compared for $n = 2$ to those of Akutsu *et al* (1987). The values given by Akutsu *et al* for $\alpha(10_{100})$ and $\alpha(10_{112})$ have incorrect factors for $1 - \alpha(A)$.

The $\{2\}su(n)_q$ polynomial distinguishes all of the pairs of knots of the $\{1\}su(n)_q$ polynomials for knots of ten or fewer crossings, both the five pairs for all n and the nine further pairs only for $n = 2$. However, the $\{2\}su(n)_q$ polynomial has four pairs for knots of ten or fewer crossings. They are: 3_1 and 7_7, 7_6 and 10_{60}, 8_{11} and 10_7, 9_{44} and 10_{71}.

The polynomial of a mirror image to a knot is obtained by substituting q^{-1} for q in the knot polynomial. Knots which are amphichiral have polynomials symmetric in q and q^{-1}. The $\{1\}su(n)_q$ polynomials for the knots 9_{42}, 10_{48}, 10_{71}, 10_{91}, 10_{104}, and 10_{125} are symmetric in q and q^{-1}, as shown in Table 7.4, but these knots are not amphichiral. The $\{2\}su(n)_q$ polynomial does correctly reflect the amphichirality/nonamphichirality of all the knots having ten or fewer crossings.

Table 7.3 summarizes the differences between the $\{1\}su(n)_q$ and $\{2\}su(n)_q$ polynomials. Table 7.4 gives the $\{1\}su(n)_q$ and $\{2\}su(n)_q$ polynomials for all knots.

Table 7.3: Summary of pairs and falsely amphichiral knots

Invariant	Knots with same polynomial	Falsely amphichiral knots
$\{1\}su(n)_q$	5_1–10_{132}, 8_8–10_{129}, 8_{16}–10_{156}, 10_{25}–10_{56}, 10_{40}–10_{103}	9_{42}, 10_{48}, 10_{71}, 10_{91}, 10_{104}, 10_{125}
$\{1\}su(2)_q$ (additional)	10_{22}–10_{35}, 10_{41}–10_{94}, 10_{43}–10_{91}, 10_{59}–10_{106}, 10_{60}–10_{83}, 10_{71}–10_{104}, 10_{73}–10_{86}, 10_{81}–10_{109}, 10_{137}–10_{155}	
$\{2\}su(n)_q$	3_1–7_7, 7_6–10_{60}, 8_{11}–10_7, 9_{44}–10_{71}	none

Table 7.4: $\{1\}su(n)_q$ and $\{2\}su(n)_q$ polynomials. The table gives for each knot of 10 or less crossings its braid index (BI), braid word, reduced $\{1\}su(n)_q$ polynomial and reduced $\{2\}su(n)_q$ polynomial. The notes column gives details of knots having the same $\{1\}su(n)_q$ polynomial or $\{2\}$ $su(n)_q$ polynomial with first the other knot with the same polynomial being given. Other notes are knots that are amphichiral (A), and knots that are not amphichiral but falsely appear to be so from their $\{1\}su(n)_q$ polynomial (FA). The two knots whose $\{2\}$ $su(2)_q$ polynomials are different to those given by Akutsu *et al* (1987) are identified with DADW.

The notation for the braid words is the same as that of Jones (1987), that is i^n represents b_i^n and $\bar{i}^n$ represents b_i^{-n}. The slashes indicate groups. For example, the braid word for the knot 7_7 is $b_1b_2^{-1}b_3b_1b_2^2b_3^{-2}b_2^{-1}$.

The reduced $\{1\}su(n)_q$ polynomial $\beta(A)$ of the knot A is tabulated below where $\beta(A) = \frac{(1-\alpha(A))}{(q^{n+1}-q^{-(n+1)})(q^{n-1}-q^{-(n-1)})}$, where $\alpha(A)$ is the $\{1\}su(n)_q$ polynomial for the knot A. The terms in brackets are coefficients of $1, q^{\pm 2}, q^{\pm 4}, \ldots$, the overlined terms being negative coefficients. For example, $\beta(7_7) = (-2+q^2+q^{-2}) - q^{2n}$.

The reduced $\{2\}su(n)_q$ polynomials $\gamma(A)$ are tabulated below where $\gamma(A) = \frac{(1-\alpha(A))}{(q^{n+2}-q^{-(n+2)})(q^{n-1}-q^{-(n-1)})}$. The terms in brackets are coefficients of $1, q^2, q^4 \ldots$ (notice the difference to the reduced $\{1\}su(n)_q$ polynomials). For example, $\gamma(7_7) = q^{2n+1} + q^{4n-1}(1+q^6) - q^{6n+5}$.

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
3_1	2	7_7, $\{2\}su(n)_q$	1^3	$q^{2n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n+5}(\bar{1})$
4_1	3	A	$1\bar{2}/1\bar{2}$	$(\bar{1})$	$q^{-2n-3}(\bar{1})+q^{-3}(1\,\bar{1}\,\bar{1}\,1)+q^{2n+3}(\bar{1})$
5_1	2	10_{132}, $\{1\}su(n)_q$	1^5	$q^{2n}(1)+q^{4n}(0\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n+1}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})$
5_2	3		$1^22^2/\bar{1}2$	$q^{2n}(1)+q^{4n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-1}(1\,1\,\bar{1}\,0\,1)+q^{8n+3}(1\,\bar{1}\,\bar{1}\,1)+q^{10n+9}(\bar{1})$
6_1	4		$\bar{1}23^2/\bar{1}2\bar{3}$	$(\bar{1})+q^{2n}(\bar{1})$	$q^{-2n-3}(\bar{1})+q^{-1}(\bar{2}\,\bar{1}\,1)+q^{2n-3}(1\,0\,\bar{2}\,0\,1)+q^{4n+1}(1\,\bar{1}\,\bar{1}\,1)+q^{6n+7}(\bar{1})$
6_2	3		$\bar{1}2/\bar{1}2^3$	$q^{2n}(1\,\bar{1})$	$q^{2n-7}(\bar{1}\,0\,0\,\bar{1}\,0\,0\,\bar{1})+q^{4n-7}(1\,\bar{1}\,\bar{1}\,3\,0\,\bar{2}\,2\,0\,\bar{1}\,1)+q^{6n-1}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
6_3	3	A	$\bar{1}2^2/\bar{1}^22$	$(\bar{1}\,1)$	$q^{-2n-9}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})+q^{-9}(1\,\bar{1}\,0\,4\,\bar{2}\,\bar{2}\,4\,0\,\bar{1}\,1)+q^{2n-3}(\bar{1}\,1\,0\,\bar{2}\,1\,1\,\bar{1})$
7_1	2		1^7	$q^{2n}(1)+q^{4n}(01)+q^{6n}(101)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-7}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{12n-9}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{14n-3}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1}\,\bar{1}\,\bar{1}\,\bar{1}\,\bar{1}\,0\,\bar{1})$
7_2	4		$\bar{1}2\bar{3}/1^223^3$	$q^{2n}(1)+q^{4n}(1)+q^{6n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-1}(1\,1\,\bar{1}\,0\,1)+q^{8n-1}(1\,1\,0\,\bar{1}\,0\,1)+q^{10n+3}(1\,0\,\bar{2}\,0\,1)+q^{12n+7}(1\,\bar{1}\,\bar{1}\,1)+q^{14n+13}(\bar{1})$
7_3	3		$1^22/\bar{1}2^4$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(01)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(1\,1\,\bar{1}\,0\,2\,0\,\bar{1}\,1\,0\,0\,1)+q^{12n-1}(1\,\bar{1}\,\bar{1}\,2\,0\,\bar{2}\,1\,0\,\bar{1}\,1)+q^{14n+5}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})$
7_4	4		$1^223^2/\bar{1}2\bar{3}/2$	$q^{2n}(1)+q^{4n}(2)+q^{6n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-1}(2\,1\,\bar{3}\,1\,2)+q^{8n-1}(1\,2\,1\,\bar{3}\,0\,3)+q^{10n+3}(2\,1\,\bar{3}\,\bar{1}\,2)+q^{12n+7}(1\,\bar{2}\,\bar{2}\,1)+q^{14n+13}(\bar{1})$
7_5	3		$1^42/\bar{1}2^2$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{1}1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(1\,1\,\bar{2}\,0\,3\,\bar{2}\,\bar{1}\,2\,\bar{1}\,\bar{1}\,1)+q^{12n-1}(1\,\bar{2}\,\bar{2}\,4\,\bar{1}\,\bar{4}\,3\,0\,\bar{2}\,1)+q^{14n+5}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
7_6	4	10_{60}, $\{2\}su(n)_q$	$1\bar{2}3/\bar{1}^22^33$	$q^{2n}(2\,\bar{1})+q^{4n}(1)$	$q^{2n-7}(\bar{1}\,1\,1\,\bar{2}\,1\,1\,\bar{1})+q^{4n-7}(1\,\bar{2}\,\bar{1}\,6\,\bar{2}\,\bar{4}\,5\,0\,\bar{2}\,1)+q^{6n-3}(\bar{1}\,\bar{1}\,4\,0\,\bar{5}\,2\,2\,\bar{2})+q^{8n+3}(2\,0\,\bar{2}\,1\,1)+q^{10n+9}(\bar{1})$
7_7	4	3_1, $\{2\}su(n)_q$	$1\bar{2}3/12^2\bar{3}^2/\bar{2}$	$(\bar{2}\,1)+q^{2n}(\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n+5}(\bar{1})$
8_1	5		$\bar{1}234^2/\bar{2}3\bar{4}/\bar{1}2$	$(\bar{1})+q^{2n}(\bar{1})+q^{4n}(\bar{1})$	$q^{-2n-3}(\bar{1})+q^{-1}(\bar{2}\bar{1}1)+q^{2n-1}(\bar{1}\bar{2}01)+q^{4n-3}(10\bar{1}\bar{1}01)+q^{6n+1}(10\bar{2}01)+q^{8n+5}(1\bar{1}\bar{1}1)+q^{10n+11}(\bar{1})$
8_2	3		$\bar{1}2^5/\bar{1}2$	$q^{2n}(1)+q^{4n}(\bar{1}\,1\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-11}(\bar{1}00\bar{1}00\bar{1}0\bar{1}\bar{1}00\bar{1})+q^{8n-11}(1\bar{1}\bar{1}30\bar{2}21\bar{1}10\bar{1}20\bar{1}1)+q^{10n-5}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1\,\bar{1}\,0\,1\,\bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
8_3	5	A	$\bar{1}^2\bar{2}3\bar{4}/1\bar{2}34^2$	$q^{-2n}(\bar{1})+(\bar{2})+q^{2n}(\bar{1})$	$q^{-6n-7}(\bar{1})+q^{-4n-5}(\bar{2}\bar{1}1)+q^{-2n-7}(11\bar{3}\bar{2}11)+q^{-5}(13\bar{4}\bar{4}31)+q^{2n-3}(11\bar{2}\bar{3}11)+q^{4n+1}(1\bar{1}\bar{2})+q^{6n+7}(\bar{1})$
8_4	4		$1^3\bar{2}\bar{3}^2/1\bar{2}3$	$(0\bar{1})+q^{2n}(1\bar{1})$	$q^{-2n-11}(\bar{1}00\bar{1}\bar{1}0\bar{1})+q^{-11}(1\bar{1}\bar{1}20\bar{2}10\bar{1}1)+q^{2n-9}(10\bar{2}12\bar{1}\bar{1}1\bar{1}01)+q^{4n-7}(1\bar{1}\bar{2}21\bar{2}00\bar{1})+q^{6n-1}(\bar{1}11\bar{2}01\bar{1})$
8_5	3		$1^3\bar{2}/1^3\bar{2}$	$q^{2n}(1)+q^{4n}(\bar{2}1\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-11}(\bar{1}00\bar{2}\bar{1}0\bar{3}\bar{1}\bar{1}\bar{2}\bar{1}0\bar{1})+q^{8n-11}(1\bar{1}\bar{1}41\bar{3}43\bar{2}32\bar{2}21\bar{1}1)+q^{10n-5}(\bar{1}11\bar{3}\bar{1}3\bar{2}\bar{2}2\bar{1}\bar{1}1\bar{1})$
8_6	4		$\bar{1}2\bar{3}/\bar{1}2^3 3^2$	$q^{2n}(1\bar{1})+q^{4n}(1\bar{1})$	$q^{2n-7}(\bar{1}00\bar{1}00\bar{1})+q^{4n-5}(\bar{2}03\bar{2}\bar{2}2\bar{1}\bar{1}1)+q^{6n-7}(10\bar{3}25\bar{4}\bar{2}4\bar{1}\bar{1}1)+q^{8n-3}(1\bar{2}\bar{1}5\bar{2}\bar{4}40\bar{2}1)+q^{10n+3}(\bar{1}11\bar{2}01\bar{1})$
8_7	3		$1^4\bar{2}^2/1\bar{2}$	$q^{2n}(2\bar{1}1)$	$q^{2n-13}(\bar{1}11\bar{2}02\bar{1}01\bar{1}01\bar{1})+q^{4n-13}(1\bar{1}04\bar{2}\bar{3}61\bar{4}41\bar{2}30\bar{1}1)+q^{6n-7}(\bar{1}10\bar{3}12\bar{4}02\bar{2}01\bar{1})$
8_8	4	10_{129}, $\{1\}su(n)_q$	$\bar{1}2\bar{3}/1^2 2^2\bar{3}^2$	$(\bar{1}1)+q^{2n}(\bar{1}1)$	$q^{-2n-9}(\bar{1}11\bar{2}01\bar{1})+q^{-7}(\bar{1}22\bar{4}04\bar{1}\bar{1}1)+q^{2n-9}(10\bar{2}42\bar{6}25\bar{2}\bar{1}1)+q^{4n-5}(1\bar{2}04\bar{4}\bar{2}5\bar{1}21)+q^{6n+1}(\bar{1}10\bar{2}11\bar{1})$
8_9	3	A	$\bar{1}2/\bar{1}^3 2^3$	$(\bar{2}1\bar{1})$	$q^{-2n-15}(\bar{1}10\bar{3}12\bar{4}02\bar{2}01\bar{1})+q^{-15}(1\bar{1}03\bar{3}\bar{1}7\bar{4}\bar{4}7\bar{1}\bar{3}30\bar{1}1)+q^{2n-9}(\bar{1}10\bar{2}20\bar{4}21\bar{3}01\bar{1})$
8_{10}	3		$\bar{1}2^2/\bar{1}^2 2^3$	$q^{2n}(3\bar{1}1)$	$q^{2n-13}(\bar{1}11\bar{3}03\bar{3}02\bar{2}\bar{1}1\bar{1})+q^{4n-13}(1\bar{1}05\bar{2}\bar{4}102\bar{6}83\bar{3}41\bar{1}1)+q^{6n-7}(\bar{1}10\bar{4}13\bar{7}\bar{1}3\bar{3}\bar{1}1\bar{1})$
8_{11}	4	10_7, $\{2\}su(n)_q$	$\bar{1}2^2\bar{3}/23^2/\bar{1}2$	$q^{2n}(2\bar{1})+q^{4n}(1\bar{1})$	$q^{2n-7}(\bar{1}11\bar{2}11\bar{1})+q^{4n-5}(\bar{3}06\bar{4}\bar{4}5\bar{1}21)+q^{6n-7}(10\bar{4}17\bar{4}\bar{6}50\bar{2}1)+q^{8n-3}(1\bar{2}\bar{1}6\bar{1}\bar{5}41\bar{2}1)+q^{10n+3}(\bar{1}11\bar{2}01\bar{1})$
8_{12}	5	A	$12\bar{3}\bar{4}^2/2\bar{3}\bar{4}/1\bar{2}34$	$q^{-2n}(\bar{1})+(\bar{3}1)+q^{2n}(\bar{1})$	$q^{-6n-7}(\bar{1})+q^{-4n-9}(11\bar{3}\bar{1}2)+q^{-2n-9}(\bar{2}34\bar{8}\bar{3}50\bar{1})+q^{-9}(1\bar{3}1\,10\,\bar{9}\bar{9}\,10\,1\bar{3}1)+q^{2n-5}(\bar{1}05\bar{3}\bar{8}43\bar{2})+q^{4n+1}(2\bar{1}\bar{3}11)+q^{6n+7}(\bar{1})$
8_{13}	4		$1^2 2\bar{3}/2\bar{3}^2/\bar{1}2$	$(\bar{2}1)+q^{2n}(\bar{1}1)$	$q^{-2n-9}(\bar{1}21\bar{4}12\bar{1})+q^{-7}(\bar{2}34\bar{8}\bar{1}7\bar{2}\bar{2}1)+q^{2n-9}(10\bar{3}44\bar{8}\bar{1}7\bar{2}\bar{2}1)+q^{4n-5}(1\bar{2}05\bar{4}\bar{3}60\bar{2}1)+q^{6n+1}(\bar{1}10\bar{2}11\bar{1})$
8_{14}	4		$1^2 2^2\bar{3}/\bar{1}2\bar{3}/2$	$q^{2n}(2\bar{1})+q^{4n}(2\bar{1})$	$q^{2n-7}(\bar{1}11\bar{2}11\bar{1})+q^{4n-5}(\bar{3}16\bar{5}\bar{3}6\bar{1}21)+q^{6n-7}(10\bar{5}39\bar{8}\bar{5}8\bar{1}\bar{3}1)+q^{8n-3}(1\bar{3}\bar{1}9\bar{4}\bar{8}71\bar{3}1)+q^{10n+3}(\bar{1}21\bar{4}12\bar{1})$
8_{15}	4		$1^2\bar{2}3^2/12^2 3$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{2}2)+q^{8n}(\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-3}(1001001)+q^{8n-5}(1001001001)+q^{10n-5}(21\bar{5}36\bar{7}06\bar{3}\bar{1}2)+q^{12n-3}(12\bar{7}\bar{1}\,12\,\bar{8}\bar{8}\,11\,\bar{2}\bar{5}3)+q^{14n+3}(\bar{3}\bar{3}7\bar{2}\bar{9}42\bar{5})+q^{16n+9}(31\bar{2}22)+q^{18n+15}(\bar{1})$
8_{16}	3	10_{156}, $\{1\}su(n)_q$	$1^3 2^3/\bar{1}^2\bar{2}^2$	$q^{2n}(3\bar{2}1)$	$q^{2n-13}(\bar{1}22\bar{5}16\bar{5}\bar{1}5\bar{3}\bar{1}2\bar{1})+q^{4n-13}(1\bar{2}\bar{1}7\bar{5}\bar{9}\,14\,1\,\overline{14}\,10\,3\bar{8}51\bar{2}1)+q^{6n-7}(\bar{1}21\bar{6}37\bar{9}\bar{1}7\bar{4}\bar{1}2\bar{1})$
8_{17}	3	A	$1^2 2^3/\bar{1}^3\bar{2}^2$	$(\bar{3}2\bar{1})$	$q^{-2n-15}(\bar{1}20\bar{6}55\,\overline{10}\,26\bar{5}02\bar{1})+q^{-15}(1\bar{2}06\bar{8}\bar{3}\,17\,\overline{11}\,\overline{11}\,17\,\bar{3}\bar{8}60\bar{2}1)+q^{2n-9}(\bar{1}20\bar{5}62\,\overline{10}\,55\bar{6}02\bar{1})$
8_{18}	3	A	$\bar{1}^2 2^2/1^2\bar{2}/1\bar{2}^2$	$(\bar{3}3\bar{1})$	$q^{-2n-15}(\bar{1}30\bar{9}87\,\overline{15}\,39\bar{8}03\bar{1})+q^{-15}(1\bar{3}09\,\overline{12}\,\bar{4}\,26\,\overline{15}\,\overline{15}\,26\,\bar{4}\,\overline{12}\,90\bar{3}1)+q^{2n-9}(\bar{1}30\bar{8}93\,\overline{15}\,78\bar{9}03\bar{1})$
8_{19}	3		$1^3 2/1^2 2^2$	$q^{2n}(1)+q^{4n}(0\ 1)+q^{6n}(101)+q^{8n}(\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-3}(1001001)+q^{8n-5}(1001001001)+q^{10n-7}(1001001001001)+q^{12n-9}(1001001001001001)+q^{14n-3}(\bar{1}\bar{1}\bar{1}\bar{1}\bar{2}\bar{1}\bar{2}\bar{2}\bar{2}\bar{1}\bar{2}\bar{1}\bar{1})+q^{16n+5}(11111111)+q^{18n+15}(\bar{1})$
8_{20}	3		$1^3 2/\bar{1}^3 2$	$q^{2n}(01)$	$q^{2n-3}(1\bar{1}010\bar{1})+q^{4n-5}(1011022001)+q^{6n+1}(\bar{1}0\bar{1}\bar{1}00\bar{1})$
8_{21}	3		$1\bar{2}^2/1^2 2^3$	$q^{2n}(1)+q^{4n}(1\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-5}(\bar{1}\bar{2}20\bar{4}01\bar{2})+q^{8n-3}(1\bar{1}14\bar{3}\bar{1}4\bar{1}\bar{1}1)+q^{10n+3}(\bar{1}10\bar{2}11\bar{1})$
9_1	2		1^9	$q^{2n}(1)+q^{4n}(0\ 1)+q^{6n}(101)+q^{8n}(0101)$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-3}(1001001)+q^{8n-5}(1001001001)+q^{10n-7}(1001001001001)+q^{12n-9}(1001001001001001)+q^{14n-11}(1001001001001001001)+q^{16n-13}(1001001001001001001001)+q^{18n-7}(\bar{1}00\bar{1}\bar{1}0\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{2}\bar{1}\bar{1}\bar{1}\bar{1}0\bar{1})$
9_2	5		$1^2 234^3/\bar{1}3\bar{4}/2\bar{3}$	$q^{2n}(1)+q^{4n}(1)+q^{6n}(1)+q^{8n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-1}(11\bar{1}01)+q^{8n-1}(110\bar{1}01)+q^{10n-1}(1100\bar{1}01)+q^{12n+3}(10\bar{1}\bar{1}01)+q^{14n+7}(10\bar{2}01)+q^{16n+11}(1\bar{1}\bar{1}1)+q^{18n+17}(\bar{1})$
9_3	3		$1\bar{2}/1^6 2^2$	$q^{2n}(1)+q^{4n}(0\ 1)+q^{6n}(101)+q^{8n}(101)$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-3}(1001001)+q^{8n-5}(1001001001)+q^{10n-7}(1001001001001)+q^{12n-9}(1001001001001001)+q^{14n-9}(11\bar{1}020\bar{1}1100001001)+q^{16n-5}(1\bar{1}\bar{1}20\bar{2}11\bar{1}00\bar{1}10\bar{1}1)+q^{18n+1}(\bar{1}00\bar{1}\bar{1}0\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}0\bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
9_4	4		$\bar{1}2/1^2 2 3^4/\bar{2}3$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(01)+q^{8n}(0\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(1\,1\,\bar{1}\,0\,2\,0\,\bar{1}\,1\,0\,0\,1)+q^{12n-5}(1\,1\,0\,\bar{1}\,1\,2\,\bar{1}\,\bar{1}\,1\,0\,0\,1)+q^{14n-1}(1\,0\,\bar{2}\,1\,2\,\bar{2}\,\bar{1}\,1\,\bar{1}\,0\,1)+q^{16n+3}(1\,\bar{1}\,\bar{1}\,2\,0\,\bar{2}\,1\,0\,\bar{1}\,1)+q^{18n+9}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})$
9_5	5		$1^2 2 3^2 4/\bar{1}2 3^2 \bar{4}/2\overline{34}$	$q^{2n}(1)+q^{4n}(2)+q^{6n}(2)+q^{8n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-1}(2\,1\,\bar{3}\,1\,2)+q^{8n-1}(2\,2\,\bar{1}\,\bar{3}\,1\,3)+q^{10n-1}(1\,2\,2\,\bar{1}\,\bar{4}\,1\,3)+q^{12n+3}(2\,2\,\bar{1}\,\bar{4}\,0\,3)+q^{14n+7}(2\,0\,\bar{4}\,\bar{1}\,2)+q^{16n+11}(1\,\bar{2}\,\bar{2}\,1)+q^{18n+17}(\bar{1})$
9_6	3		$1^2 2^2/1^5\bar{2}$	$q^{2n}(1)+q^{4n}(0\;\;1)+q^{6n}(1\,0\,1)+q^{8n}(1\,\bar{1}\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-7}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{12n-9}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{14n-9}(1\,1\,\bar{2}\,0\,3\,\bar{2}\,\bar{2}\,2\,0\,\bar{2}\,1\,\bar{1}\,\bar{1}\,1\,\bar{1}\,\bar{1}\,1)+q^{16n-5}(1\,\bar{2}\,\bar{2}\,4\,\bar{1}\,\bar{5}\,3\,2\,\bar{4}\,1\,1\,\bar{3}\,2\,0\,\bar{2}\,1)+q^{18n+1}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1\,\bar{1}\,0\,1\,\bar{1})$
9_7	4		$1^3 2 3^2/\bar{1}2^3\bar{3}$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{1}1)+q^{8n}(\bar{1}\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(1\,1\,\bar{2}\,0\,3\,\bar{2}\,\bar{1}\,2\,\bar{1}\,\bar{1}\,1)+q^{12n-5}(1\,1\,\bar{1}\,\bar{2}\,2\,3\,\bar{3}\,\bar{1}\,3\,\bar{1}\,\bar{1}\,1)+q^{14n-1}(1\,\bar{1}\,\bar{4}\,3\,4\,\bar{6}\,\bar{1}\,4\,\bar{2}\,\bar{1}\,1)+q^{16n+3}(1\,\bar{2}\,\bar{1}\,5\,\bar{2}\,\bar{4}\,4\,0\,\bar{2}\,1)+q^{18n+9}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
9_8	5		$1\bar{2}3\bar{4}/\bar{1}^2 34/\bar{2}34^2$	$(1\,\bar{1})+q^{2n}(2\,\bar{1})+q^{4n}(1)$	$q^{-2n-11}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})+q^{-11}(1\,\bar{2}\,\bar{1}\,5\,\bar{2}\,\bar{4}\,4\,0\,\bar{2}\,1)+q^{2n-9}(1\,\bar{1}\,\bar{3}\,4\,3\,\bar{5}\,0\,4\,\bar{2}\,\bar{1}\,1)+q^{4n-7}(1\,\bar{2}\,\bar{3}\,5\,2\,\bar{5}\,0\,2\,\bar{1}\,\bar{1})+q^{6n-3}(\bar{1}\,\bar{1}\,4\,1\,\bar{4}\,1\,2\,\bar{1})+q^{8n+3}(2\,0\,\bar{2}\,1\,1)+q^{10n+9}(\bar{1})$
9_9	3		$1^3\bar{2}/1^4 2^2$	$q^{2n}(1)+q^{4n}(0\;\;1)+q^{6n}(1\,0\,1)+q^{8n}(2\,\bar{1}\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-7}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{12n-9}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{14n-9}(1\,1\,\bar{2}\,0\,4\,\bar{2}\,\bar{2}\,4\,\bar{1}\,\bar{2}\,3\,\bar{1}\,\bar{2}\,2\,0\,\bar{1}\,1)+q^{16n-5}(1\,\bar{2}\,\bar{2}\,5\,\bar{1}\,\bar{7}\,5\,2\,\bar{7}\,3\,2\,\bar{5}\,2\,1\,\bar{2}\,1)+q^{18n+1}(\bar{1}\,1\,1\,\bar{3}\,0\,3\,\bar{3}\,\bar{1}\,2\,\bar{2}\,\bar{1}\,1\,\bar{1})$
9_{10}	4		$\bar{1}2/1^2 2^3\bar{3}/23^2$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{1}2)+q^{8n}(0\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(2\,1\,\bar{4}\,2\,5\,\bar{4}\,\bar{1}\,4\,\bar{2}\,0\,2)+q^{12n-5}(1\,2\,1\,\bar{5}\,1\,9\,\bar{5}\,\bar{5}\,7\,\bar{1}\,\bar{2}\,3)+q^{14n-1}(2\,1\,\bar{4}\,0\,7\,\bar{2}\,\bar{5}\,4\,0\,\bar{2}\,2)+q^{16n+3}(1\,\bar{2}\,\bar{2}\,3\,0\,\bar{4}\,1\,0\,\bar{2}\,1)+q^{18n+9}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})$
9_{11}	4		$\bar{1}2\bar{3}/23/\bar{1}23^4$	$q^{2n}(1)+q^{4n}(\bar{1}2\bar{1})+q^{6n}(01)$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-11}(\bar{1}\,1\,1\,\bar{2}\,1\,2\,\bar{1}\,0\,1\,\bar{1}\,1\,1\,\bar{1})+q^{8n-11}(1\,\bar{2}\,\bar{1}\,6\,\bar{2}\,\bar{5}\,6\,2\,\bar{5}\,3\,1\,\bar{3}\,4\,0\,\bar{2}\,1)+q^{10n-7}(\bar{1}\,\bar{1}\,4\,0\,\bar{6}\,3\,5\,\bar{5}\,\bar{1}\,3\,\bar{3}\,1\,2\,\bar{2})+q^{12n-1}(2\,0\,\bar{2}\,2\,2\,\bar{2}\,0\,1\,\bar{1}\,1\,1)+q^{14n+5}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})$
9_{12}	5		$1\bar{2}34^2/\bar{1}^2 2^3 3\bar{4}$	$q^{2n}(2\bar{1})+q^{4n}(2\bar{1})+q^{6n}(1)$	$q^{2n-7}(\bar{1}\,1\,1\,\bar{2}\,1\,1\,\bar{1})+q^{4n-5}(\bar{2}\,1\,5\,\bar{4}\,\bar{3}\,5\,\bar{1}\,\bar{2}\,1)+q^{6n-7}(1\,\bar{1}\,\bar{4}\,5\,7\,\bar{8}\,\bar{3}\,7\,\bar{1}\,\bar{2}\,1)+q^{8n-1}(\bar{3}\,2\,8\,\bar{7}\,\bar{6}\,8\,0\,\bar{3}\,1)+q^{10n+1}(\bar{1}\,0\,4\,\bar{2}\,\bar{5}\,3\,2\,\bar{2})+q^{12n+7}(2\,0\,\bar{2}\,1\,1)+q^{14n+13}(\bar{1})$
9_{13}	4		$1^2 2\bar{3}/23^2/\bar{1}2^3$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{1}2)+q^{8n}(\bar{1}\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(2\,1\,\bar{4}\,2\,5\,\bar{4}\,\bar{1}\,4\,\bar{2}\,0\,2)+q^{12n-5}(1\,2\,0\,\bar{5}\,3\,8\,\bar{6}\,\bar{3}\,7\,\bar{2}\,\bar{2}\,3)+q^{14n-1}(2\,0\,\bar{6}\,2\,8\,\bar{7}\,\bar{4}\,6\,\bar{2}\,\bar{3}\,2)+q^{16n+3}(1\,\bar{3}\,\bar{2}\,6\,\bar{2}\,\bar{6}\,4\,0\,\bar{3}\,1)+q^{18n+9}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
9_{14}	5		$12\bar{3}/2\overline{34}/\bar{1}2^2 34^2/\overline{23}$	$(\bar{2}\,1)+q^{2n}(\bar{2}\,1)+q^{4n}(\bar{1})$	$q^{-2n-9}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})+q^{-7}(\bar{1}\,3\,2\,\bar{8}\,0\,7\,\bar{2}\,\bar{2}\,1)+q^{2n-9}(1\,\bar{1}\,\bar{2}\,6\,0\,\overline{11}\,3\,8\,\bar{3}\,\bar{2}\,1)+q^{4n-3}(\bar{2}\,3\,3\,\overline{10}\,\bar{1}\,9\,\bar{2}\,\bar{3}\,1)+q^{6n-1}(\bar{1}\,1\,3\,\bar{5}\,\bar{3}\,5\,1\,\bar{2})+q^{8n+5}(2\,\bar{1}\,\bar{2}\,2\,1)+q^{10n+11}(\bar{1})$
9_{15}	5		$1\bar{2}34/1\overline{23}4^2/3$	$q^{2n}(2\bar{1})+q^{4n}(3\bar{1})+q^{6n}(1)$	$q^{2n-7}(\bar{1}\,1\,1\,\bar{2}\,1\,1\,\bar{1})+q^{4n-5}(\bar{2}\,2\,5\,\bar{5}\,\bar{2}\,6\,\bar{1}\,\bar{2}\,1)+q^{6n-7}(1\,\bar{1}\,\bar{5}\,7\,8\,\overline{13}\,\bar{2}\,10\,\bar{2}\,\bar{3}\,1)+q^{8n-1}(\bar{4}\,3\,11\,\overline{11}\,\bar{8}\,12\,0\,\bar{4}\,1)+q^{10n+1}(\bar{1}\,1\,6\,\bar{3}\,\bar{7}\,5\,3\,\bar{2})+q^{12n+7}(2\,\bar{1}\,\bar{3}\,1\,1)+q^{14n+13}(\bar{1})$
9_{16}	3		$1^2\bar{2}/1^3 2^4$	$q^{2n}(1)+q^{4n}(0\;\;1)+q^{6n}(1\,0\,1)+q^{8n}(2\,\bar{2}\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-7}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{12n-9}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{14n-9}(1\,1\,\bar{3}\,0\,5\,\bar{5}\,\bar{3}\,6\,\bar{4}\,\bar{5}\,5\,\bar{3}\,\bar{4}\,2\,\bar{1}\,\bar{2}\,1)+q^{16n-5}(1\,\bar{3}\,\bar{3}\,8\,\bar{2}\,\overline{11}\,10\,4\,\overline{12}\,7\,5\,\bar{8}\,4\,2\,\bar{3}\,1)+q^{18n+1}(\bar{1}\,2\,2\,\bar{5}\,1\,6\,\bar{5}\,\bar{2}\,5\,\bar{3}\,\bar{1}\,2\,\bar{1})$
9_{17}	4		$1^3\overline{23}/1\overline{23}/1\bar{2}3$	$(0\,\bar{1})+q^{2n}(2\,\bar{2}\,1)$	$q^{-2n-11}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})+q^{-13}(1\,2\,\bar{2}\,\bar{1}\,4\,0\,\bar{3}\,2\,0\,\bar{1}\,2)+q^{2n-13}(\bar{2}\,1\,4\,\bar{4}\,\bar{5}\,6\,2\,\bar{6}\,1\,1\,\bar{3}\,2\,0\,\bar{1})+q^{4n-13}(1\,\bar{2}\,\bar{1}\,7\,\bar{3}\,\bar{9}\,8\,5\,\bar{9}\,2\,3\,\bar{5}\,3\,1\,\bar{2}\,1)+q^{6n-7}(\bar{1}\,2\,1\,\bar{5}\,2\,5\,\bar{5}\,\bar{1}\,4\,\bar{3}\,0\,2\,\bar{1})$
9_{18}	4		$1^2 2/\bar{1}2^2 3^2/2^2\bar{3}$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{2}2)+q^{8n}(\bar{1}\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(2\,1\,\bar{5}\,3\,6\,\bar{7}\,0\,6\,\bar{3}\,\bar{1}\,2)+q^{12n-5}(1\,2\,0\,\bar{7}\,3\,11\,\overline{10}\,\bar{5}\,11\,\bar{3}\,\bar{4}\,3)+q^{14n-1}(2\,0\,\bar{7}\,2\,11\,\bar{8}\,\bar{7}\,9\,\bar{1}\,\bar{4}\,2)+q^{16n+3}(1\,\bar{3}\,\bar{2}\,7\,\bar{2}\,\bar{7}\,5\,1\,\bar{3}\,1)+q^{18n+9}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
9_{19}	5		$1^2 2\bar{3}\bar{4}/\bar{1}2\bar{3}4/2\bar{3}\bar{4}$	$q^{-2n}(\bar{1}) + (\bar{3}\ 1) + q^{2n}(\bar{2}\ 1)$	$q^{-6n-7}(\bar{1}) + q^{-4n-9}(1\ 1\ \bar{3}\ \bar{1}\ 2) + q^{-2n-9}(\bar{1}\ 4\ 3\ \bar{8}\ \bar{2}\ 5\ 0\ \bar{1}) + q^{-9}(\bar{1}\ \bar{3}\ 6\ 7\ \overline{14}\ \bar{5}\ 11\ \bar{1}\ \bar{3}\ 1) + q^{2n-9}(1\ \bar{1}\ \bar{5}\ 7\ 8\ \overline{14}\ \bar{5}\ 11\ \bar{1}\ \bar{3}\ 1) + q^{4n-5}(1\ \bar{3}\ 0\ 9\ \bar{6}\ \bar{8}\ 8\ 1\ \bar{3}\ 1) + q^{6n+1}(\bar{1}\ 2\ 1\ \bar{4}\ 1\ 2\ \bar{1})$
9_{20}	4		$2^2\bar{3}/1^2\bar{2}3/12^2\bar{3}$	$q^{2n}(1)+q^{4n}(\bar{2}2\bar{1})+q^{6n}(\bar{1}1)$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-11}(\bar{1}\ 1\ 1\ \bar{3}\ 1\ 3\ \bar{3}\ 0\ 2\ \bar{2}\ 0\ 1\ \bar{1})+q^{8n-11}(1\ \bar{2}\ \bar{1}\ 7\ \bar{3}\ \bar{7}\ 10\ 1\ \bar{9}\ 7\ 2\ \bar{6}\ 4\ 1\ \bar{2}\ 1)+q^{10n-7}(\bar{1}\ \bar{1}\ 5\ \bar{1}\ \bar{9}\ 6\ 6\ \overline{10}\ 1\ 6\ \bar{6}\ 0\ 3\ \bar{2})+q^{12n-1}(2\ \bar{1}\ \bar{4}\ 4\ 3\ \bar{5}\ 1\ 3\ \bar{2}\ 0\ 1)+q^{14n+5}(\bar{1}\ 1\ 1\ \bar{2}\ 0\ 1\ \bar{1})$
9_{21}	5		$\bar{1}^2\bar{2}34^2/12^2 3\bar{4}/3$	$q^{2n}(3\bar{1})+q^{4n}(3\bar{1})+q^{6n}(1)$	$q^{2n-7}(\bar{1}\ 2\ 1\ \bar{4}\ 2\ 2\ \bar{1}) + q^{4n-5}(\bar{3}\ 3\ 8\ \bar{9}\ \bar{4}\ 10\ \bar{1}\ \bar{3}\ 1) + q^{6n-7}(1\ \bar{1}\ \bar{6}\ 7\ 12\ \overline{14}\ \bar{7}\ 13\ 0\ \bar{4}\ 1) + q^{8n-1}(\bar{4}\ 3\ 13\ \overline{10}\ \overline{11}\ 12\ 2\ \bar{4}\ 1) + q^{10n+1}(\bar{1}\ 1\ 6\ \bar{3}\ \bar{8}\ 4\ 3\ \bar{2}) + q^{12n+7}(2\ \bar{1}\ \bar{3}\ 1\ 1) + q^{14n+13}(\bar{1})$
9_{22}	4		$\bar{1}\bar{2}^2 3^4/2/\bar{1}2\bar{3}$	$(0\ \bar{1}) + q^{2n}(3\ \bar{2}\ 1)$	$q^{-2n-11}(\bar{1}\ 0\ 0\ \bar{1}\ \bar{1}\ 0\ \bar{1}) + q^{-13}(1\ 2\ \bar{2}\ 0\ 5\ 0\ \bar{2}\ 3\ 0\ \bar{1}\ 2) + q^{2n-13}(\bar{2}\ 1\ 4\ \bar{6}\ \bar{6}\ 8\ \bar{1}\ \bar{8}\ 3\ 0\ \bar{5}\ 2\ 0\ \bar{1}) + q^{4n-13}(1\ \bar{2}\ \bar{1}\ 8\ \bar{3}\ \overline{11}\ 12\ 7\ \overline{13}\ 5\ 6\ \bar{7}\ 3\ 2\ \bar{2}\ 1) + q^{6n-7}(\bar{1}\ 2\ 1\ \bar{6}\ 2\ 7\ \bar{7}\ \bar{2}\ 6\ \bar{3}\ \bar{1}\ 2\ \bar{1})$
9_{23}	4		$\bar{1}2/1^3 23^3/2\bar{3}$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{2}2)+q^{8n}(\bar{2}\ 1)$	$q^{2n+1}(1) + q^{4n-1}(1\ 0\ 0\ 1) + q^{6n-3}(1\ 0\ 0\ 1\ 0\ 0\ 1) + q^{8n-5}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1) + q^{10n-5}(2\ 1\ \bar{5}\ 3\ 6\ \bar{7}\ 0\ 6\ \bar{3}\ \bar{1}\ 2)+q^{12n-5}(1\ 2\ \bar{1}\ \bar{7}\ 5\ 9\ \overline{12}\ \bar{3}\ 11\ \bar{4}\ \bar{4}\ 3)+q^{14n-1}(2\ \bar{1}\ \bar{9}\ 5\ 12\ \overline{14}\ \bar{5}\ 12\ \bar{3}\ \bar{5}\ 2)+q^{16n+3}(1\ \bar{4}\ \bar{1}\ 11\ \bar{5}\ \bar{9}\ 9\ 1\ \bar{4}\ 1)+q^{18n+9}(\bar{1}\ 2\ 1\ \bar{4}\ 1\ 2\ \bar{1})$
9_{24}	4		$1\bar{2}3/1\bar{2}^3 3^2$	$(\bar{3}\ 2\ \bar{1}) + q^{2n}(0\ 1)$	$q^{-2n-15}(\bar{1}\ 2\ 0\ \bar{5}\ 5\ 3\ \bar{9}\ 3\ 5\ \bar{5}\ 0\ 2\ \bar{1}) + q^{-15}(1\ \bar{2}\ 1\ 5\ \bar{9}\ 2\ 15\ \overline{15}\ \bar{5}\ 18\ \bar{6}\ \bar{7}\ 7\ 0\ \bar{2}\ 1) + q^{2n-11}(\bar{1}\ \bar{1}\ 3\ \bar{4}\ \bar{3}\ 9\ \bar{6}\ \overline{10}\ 11\ 0\ \bar{9}\ 2\ 2\ \bar{2}) + q^{4n-5}(2\ 0\ 0\ 4\ 0\ \bar{1}\ 5\ 2\ \bar{2}\ 1\ 1) + q^{6n+1}(\bar{1}\ 0\ \bar{1}\ \bar{1}\ 0\ 0\ \bar{1})$
9_{25}	5		$\bar{1}23^2\bar{4}/\bar{1}234^2/2\bar{3}$	$q^{2n}(2\bar{1})+q^{4n}(3\bar{2})+q^{6n}(1)$	$q^{2n-7}(\bar{1}\ 1\ 1\ \bar{2}\ 1\ 1\ \bar{1}) + q^{4n-7}(\bar{1}\ \bar{3}\ 3\ 5\ \bar{7}\ \bar{2}\ 6\ \bar{2}\ \bar{2}\ 1) + q^{6n-7}(2\ \bar{2}\ \bar{8}\ 11\ 10\ \overline{18}\ \bar{1}\ 13\ \bar{5}\ \bar{3}\ 2) + q^{8n-5}(1\ 1\ \bar{8}\ 5\ 17\ \overline{16}\ \bar{9}\ 18\ \bar{2}\ \bar{6}\ 3) + q^{10n+1}(\bar{3}\ \bar{1}\ 9\ \bar{5}\ \overline{11}\ 7\ 3\ \bar{5}) + q^{12n+7}(3\ 0\ \bar{3}\ 2\ 2) + q^{14n+13}(\bar{1})$
9_{26}	4		$\bar{1}23^3/\bar{1}2\bar{3}/\bar{1}23$	$q^{2n}(3\ \bar{2}\ 1) + q^{4n}(1\ \bar{1})$	$q^{2n-13}(\bar{1}\ 2\ 1\ \bar{5}\ 2\ 5\ \bar{5}\ 0\ 4\ \bar{3}\ 0\ 2\ \bar{1}) + q^{4n-13}(1\ \bar{2}\ 1\ 6\ \bar{9}\ \bar{4}\ 17\ \bar{6}\ \overline{12}\ 13\ \bar{1}\ \bar{7}\ 6\ 0\ \bar{2}\ 1) + q^{6n-9}(\bar{1}\ 0\ 4\ \bar{6}\ \bar{6}\ 13\ \bar{1}\ \overline{15}\ 8\ 4\ \bar{8}\ 2\ 2\ \bar{2}) + q^{8n-3}(2\ \bar{2}\ \bar{3}\ 7\ 1\ \bar{7}\ 4\ 3\ \bar{3}\ 1\ 1) + q^{10n+3}(\bar{1}\ 1\ 1\ \bar{2}\ 0\ 1\ \bar{1})$
9_{27}	4		$\bar{1}2\bar{3}/\bar{1}^2 23/\bar{1}23^2$	$(\bar{3}\ 2\ \bar{1}) + q^{2n}(\bar{1}\ 1)$	$q^{-2n-15}(\bar{1}\ 2\ 0\ \bar{5}\ 5\ 3\ \bar{9}\ 3\ 5\ \bar{5}\ 0\ 2\ \bar{1}) + q^{-15}(1\ \bar{2}\ 1\ 4\ \bar{9}\ 4\ 13\ \overline{17}\ \bar{3}\ 18\ \bar{7}\ \bar{7}\ 7\ 0\ \bar{2}\ 1) + q^{2n-11}(\bar{1}\ 0\ 3\ \bar{6}\ 1\ 11\ \overline{12}\ \bar{7}\ 15\ \bar{2}\ \bar{9}\ 3\ 2\ \bar{2}) + q^{4n-5}(2\ \bar{2}\ \bar{1}\ 7\ \bar{4}\ \bar{5}\ 8\ 1\ \bar{4}\ 1\ 1) + q^{6n+1}(\bar{1}\ 1\ 0\ \bar{2}\ 1\ 1\ \bar{1})$
9_{28}	4		$1^2\bar{2}^2 3/1\bar{2}3^2$	$q^{2n}(4\ \bar{2}\ 1) + q^{4n}(1\ \bar{1})$	$q^{2n-13}(\bar{1}\ 2\ 1\ \bar{6}\ 3\ 6\ \bar{8}\ 1\ 6\ \bar{4}\ \bar{1}\ 2\ \bar{1})+q^{4n-13}(1\ \bar{2}\ 1\ 8\ \overline{10}\ \bar{5}\ 23\ \bar{8}\ \overline{15}\ 20\ 0\ \overline{10}\ 7\ 1\ \bar{2}\ 1)+q^{6n-9}(\bar{1}\ \bar{1}\ 4\ \bar{6}\ \bar{9}\ 14\ 0\ \overline{21}\ 9\ 7\ \overline{11}\ 1\ 3\ \bar{2})+q^{8n-3}(2\ \bar{1}\ \bar{2}\ 7\ 1\ \bar{7}\ 4\ 4\ \bar{3}\ 0\ 1)+q^{10n+3}(\bar{1}\ 1\ 0\ \bar{2}\ 1\ 1\ \bar{1})$
9_{29}	4		$\bar{1}2\bar{3}/2/\bar{1}2\bar{3}/2^2$	$(1\ \bar{1}) + q^{2n}(4\ \bar{2}\ 1)$	$q^{-2n-11}(\bar{1}\ 1\ 1\ \bar{2}\ 0\ 1\ \bar{1})+q^{-13}(1\ 2\ \bar{4}\ \bar{1}\ 9\ \bar{3}\ \bar{6}\ 7\ 0\ \bar{3}\ 2)+q^{2n-13}(\bar{2}\ 1\ 5\ \bar{7}\ \bar{9}\ 13\ 2\ \overline{15}\ 5\ 5\ \bar{8}\ 1\ 1\ \bar{1})+q^{4n-13}(1\ \bar{2}\ \bar{1}\ 9\ \bar{3}\ \overline{14}\ 15\ 12\ \overline{17}\ 4\ 11\ \bar{7}\ 2\ 3\ \bar{2}\ 1)+q^{6n-7}(\bar{1}\ 2\ 1\ \bar{7}\ 2\ 9\ \bar{9}\ \bar{4}\ 7\ \bar{3}\ \bar{2}\ 2\ \bar{1})$
9_{30}	4		$12\bar{3}/\bar{1}2\bar{3}/1^2 2\bar{3}^2$	$(\bar{4}\ 2\ \bar{1}) + q^{2n}(\bar{1}\ 1)$	$q^{-2n-15}(\bar{1}\ 2\ \bar{1}\ \bar{6}\ 7\ 3\ \overline{13}\ 4\ 6\ \bar{6}\ 0\ 2\ \bar{1})+q^{-15}(1\ \bar{2}\ 2\ 5\ \overline{11}\ 6\ 19\ \overline{22}\ \bar{5}\ 24\ \bar{8}\ \bar{8}\ 8\ 0\ \bar{2}\ 1)+q^{2n-11}(\bar{1}\ 0\ 3\ \bar{8}\ \bar{1}\ 14\ \overline{13}\ \overline{13}\ 17\ \bar{1}\ \overline{11}\ 3\ 2\ \bar{2})+q^{4n-5}(2\ \bar{2}\ \bar{1}\ 8\ \bar{3}\ \bar{6}\ 8\ 2\ \bar{4}\ 1\ 1)+q^{6n+1}(\bar{1}\ 1\ 0\ \bar{2}\ 1\ 1\ \bar{1})$
9_{31}	4		$\bar{1}^2 2^3 3^2/1\bar{2}3/\bar{2}$	$q^{2n}(4\ \bar{2}\ 1) + q^{4n}(2\ \bar{1})$	$q^{2n-13}(\bar{1}\ 2\ 1\ \bar{6}\ 3\ 6\ \bar{8}\ 1\ 6\ \bar{4}\ \bar{1}\ 2\ \bar{1})+q^{4n-13}(1\ \bar{2}\ 1\ 7\ \overline{11}\ \bar{3}\ 23\ \overline{11}\ \overline{14}\ 21\ \bar{1}\ \overline{10}\ 7\ 1\ \bar{2}\ 1)+q^{6n-9}(\bar{1}\ 0\ 5\ \bar{8}\ \bar{6}\ 20\ \bar{5}\ \overline{21}\ 15\ 6\ \overline{12}\ 2\ 3\ \bar{2})+q^{8n-3}(2\ \bar{3}\ \bar{4}\ 11\ \bar{1}\ \overline{13}\ 7\ 5\ \bar{5}\ 0\ 1)+q^{10n+3}(\bar{1}\ 2\ 1\ \bar{4}\ 1\ 2\ \bar{1})$
9_{32}	4		$1\bar{2}3/\bar{2}3^2/\bar{1}^2 2^2 3$	$q^{2n}(4\ \bar{3}\ 1) + q^{4n}(1\ \bar{1})$	$q^{2n-11}(1\ 0\ \bar{2}\ 0\ 3\ \bar{1}\ \bar{2}\ 2\ 0\ \bar{1}\ 1)+q^{4n-9}(1\ \bar{1}\ \bar{2}\ 0\ 2\ 1\ \bar{3}\ \bar{1}\ 1)+q^{6n-7}(1\ 0\ \bar{3}\ 2\ 3\ \bar{1}\ \bar{1}\ 0\ 0\ 1)+q^{8n-3}(1\ \bar{2}\ \bar{1}\ 4\ \bar{1}\ \bar{3}\ 2\ 0\ \bar{1}\ 1)+q^{10n+3}(\bar{1}\ 1\ 1\ \bar{2}\ 0\ 1\ \bar{1})$
9_{33}	4		$\bar{1}^3 2/\bar{1}2^2 3/1\bar{2}3$	$(\bar{4}\ 3\ \bar{1}) + q^{2n}(\bar{1}\ 1)$	$q^{-2n-15}(\bar{1}\ 3\ \bar{1}\ \bar{8}\ 11\ 4\ \overline{17}\ 7\ 9\ \bar{9}\ 0\ 3\ \bar{1})+q^{-15}(1\ \bar{3}\ 2\ 7\ \overline{17}\ 6\ 26\ \overline{31}\ \bar{8}\ 33\ \overline{12}\ \overline{13}\ 11\ 0\ \bar{3}\ 1)+q^{2n-11}(\bar{1}\ 0\ 5\ \bar{8}\ \bar{3}\ 21\ \overline{13}\ \overline{18}\ 24\ 3\ \overline{15}\ 4\ 4\ \bar{2})+q^{4n-5}(2\ \bar{2}\ \bar{2}\ 8\ \bar{3}\ \bar{8}\ 8\ 3\ \bar{5}\ 0\ 1)+q^{6n+1}(\bar{1}\ 1\ 0\ \bar{2}\ 1\ 1\ \bar{1})$
9_{34}	4		$1\bar{2}3/\bar{2}/1\bar{2}3/1\bar{2}$	$(\bar{5}\ 3\ \bar{1}) + q^{2n}(\bar{2}\ 1)$	$q^{-2n-15}(\bar{1}\ 3\ \bar{2}\ \bar{8}\ 14\ 3\ \overline{21}\ 9\ 10\ \overline{10}\ 0\ 3\ \bar{1})+q^{-15}(1\ \bar{3}\ 3\ 6\ \overline{20}\ 12\ 29\ \overline{42}\ \bar{7}\ 40\ \overline{15}\ \overline{14}\ 12\ 0\ \bar{3}\ 1)+q^{2n-11}(\bar{1}\ 1\ 5\ \overline{13}\ 0\ 29\ \overline{23}\ \overline{23}\ 33\ 2\ \overline{18}\ 5\ 4\ \bar{2})+q^{4n-5}(2\ \bar{4}\ \bar{3}\ 14\ \bar{5}\ \overline{15}\ 12\ 5\ \bar{7}\ 0\ 1)+q^{6n+1}(\bar{1}\ 2\ 1\ \bar{4}\ 1\ 2\ \bar{1})$
9_{35}	5		$1^2 23^2/\bar{1}2\bar{3}4/\bar{3}/23^2\bar{4}$	$q^{2n}(1) + q^{4n}(2) + q^{6n}(3) + q^{8n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-1}(2\ 1\ \bar{3}\ 1\ 2)+q^{8n-1}(3\ 2\ \bar{3}\ \bar{3}\ 2\ 3)+q^{10n-1}(1\ 3\ 3\ \bar{3}\ \bar{6}\ 3\ 4)+q^{12n+3}(3\ 4\ \bar{1}\ \bar{6}\ 1\ 5)+q^{14n+7}(3\ 0\ \bar{6}\ \bar{2}\ 3)+q^{16n+11}(1\ \bar{3}\ \bar{3}\ 1)+q^{18n+17}(\bar{1})$
9_{36}	4		$\bar{1}2^3\bar{3}/23^2/\bar{1}23$	$q^{2n}(1)+q^{4n}(\bar{2}2\bar{1})+q^{6n}(01)$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-11}(\bar{1}\ 1\ 1\ \bar{3}\ 1\ 3\ \bar{3}\ 0\ 2\ \bar{2}\ 0\ 1\ \bar{1})+q^{8n-11}(1\ \bar{2}\ \bar{1}\ 7\ \bar{2}\ \bar{7}\ 9\ 3\ \bar{8}\ 6\ 3\ \bar{5}\ 4\ 1\ \bar{2}\ 1)+q^{10n-7}(\bar{1}\ \bar{1}\ 4\ 0\ \bar{8}\ 2\ 7\ \bar{8}\ \bar{3}\ 5\ \bar{4}\ \bar{1}\ 2\ \bar{2})+q^{12n-1}(2\ 0\ \bar{2}\ 2\ 3\ \bar{1}\ 0\ 2\ 0\ 1\ 1)+q^{14n+5}(\bar{1}\ 0\ 0\ \bar{1}\ \bar{1}\ 0\ \bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
9_{37}	5		$\overline{12}^2 3\bar{4}/\bar{2}34^2/\overline{12}3/12$	$q^{-2n}(\bar{1}) + (\bar{4}\,1) + q^{2n}(\bar{2}\,1)$	$q^{-6n-7}(\bar{1}) + q^{-4n-9}(1\,0\,\bar{4}\,\bar{1}\,2) + q^{-2n-9}(\bar{1}\,6\,3\,\overline{12}\,\bar{1}\,7\,0\,\bar{1}) + q^{-9}(\bar{1}\,\bar{4}\,8\,11\,\overline{19}\,\bar{7}\,16\,\bar{1}\,\bar{4}\,1) + q^{2n-9}(1\,\bar{1}\,\bar{6}\,7\,11\,\overline{16}\,\overline{10}\,13\,0\,\bar{4}\,1) + q^{4n-5}(1\,\bar{3}\,0\,10\,\bar{6}\,\overline{10}\,8\,2\,\bar{3}\,1) + q^{6n+1}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})$
9_{38}	4		$\bar{1}2^3 3/1\overline{23}/1^2 23^2$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{3}3)+q^{8n}(\bar{2}\,1)$	$q^{2n+1}(1) + q^{4n-1}(1\,0\,0\,1) + q^{6n-3}(1\,0\,0\,1\,0\,0\,1) + q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1) + q^{10n-5}(3\,0\,\bar{8}\,8\,\overline{14}\,2\,10\,\bar{6}\,\bar{1}\,3) + q^{12n-5}(1\,4\,0\,\overline{14}\,9\,21\,\overline{23}\,\bar{7}\,23\,\bar{7}\,\bar{7}\,6) + q^{14n-1}(3\,0\,\overline{14}\,2\,23\,\overline{17}\,\overline{17}\,19\,\bar{1}\,\overline{10}\,3) + q^{16n+3}(1\,\bar{5}\,\bar{3}\,14\,\bar{2}\,\overline{14}\,9\,4\,\bar{5}\,1) + q^{18n+9}(\bar{1}\,2\,2\,\bar{4}\,0\,2\,\bar{1})$
9_{39}	5		$1\bar{2}3^2 4/\bar{1}^2 2^2 34/\overline{23}4$	$q^{2n}(3\,\bar{1})+q^{4n}(4\,\bar{2})+q^{6n}(1)$	$q^{2n-7}(\bar{1}\,2\,1\,\bar{4}\,2\,2\,\bar{1}) + q^{4n-7}(\bar{1}\,\bar{3}\,6\,7\,\overline{12}\,\bar{2}\,11\,\bar{2}\,\bar{3}\,1) + q^{6n-7}(2\,\bar{3}\,\overline{11}\,15\,15\,\overline{27}\,\bar{4}\,20\,\bar{5}\,\bar{5}\,2) + q^{8n-5}(1\,1\,\overline{10}\,6\,25\,\overline{21}\,\overline{17}\,24\,0\,\bar{8}\,3) + q^{10n+1}(\bar{3}\,\bar{1}\,12\,\bar{4}\,\overline{16}\,8\,6\,\bar{5}) + q^{12n+7}(3\,0\,\bar{4}\,1\,2) + q^{14n+13}(\bar{1})$
9_{40}	4		$1\bar{2}3/1\bar{2}3/1\bar{2}3$	$q^{2n}(5\,\bar{4}\,1) + q^{4n}(2\,\bar{1})$	$q^{2n-13}(\bar{1}\,4\,0\,\overline{14}\,10\,13\,\overline{22}\,1\,15\,\overline{10}\,\bar{2}\,4\,\bar{1}) + q^{4n-13}(1\,\bar{4}\,2\,15\,\overline{24}\,\overline{11}\,52\,\overline{22}\,\overline{39}\,45\,0\,\overline{25}\,13\,2\,\bar{4}\,1) + q^{6n-9}(\bar{1}\,0\,9\,\bar{9}\,\overline{20}\,34\,9\,\overline{47}\,18\,24\,\overline{23}\,0\,7\,\bar{2}) + q^{8n-3}(2\,\bar{3}\,\bar{6}\,11\,5\,\overline{16}\,2\,10\,\bar{4}\,\bar{2}\,1) + q^{10n+3}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})$
9_{41}	5		$\bar{1}2^2 3\bar{4}/123\bar{4}/\overline{12}3/2\bar{4}$	$(\bar{2}\,1) + q^{2n}(\bar{3}\,2) + q^{4n}(\bar{1})$	$q^{-2n-9}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1}) + q^{-9}(\bar{1}\,0\,5\,\bar{1}\,\bar{9}\,3\,7\,\bar{3}\,\bar{2}\,1) + q^{2n-9}(2\,\bar{3}\,\bar{3}\,14\,\bar{4}\,\overline{19}\,11\,11\,\bar{8}\,\bar{2}\,2) + q^{4n-7}(1\,0\,\bar{6}\,9\,7\,\overline{21}\,2\,18\,\bar{6}\,\bar{5}\,3) + q^{6n-1}(\bar{3}\,1\,7\,\overline{10}\,\bar{7}\,10\,1\,\bar{5}) + q^{8n+5}(3\,\bar{1}\,\bar{3}\,3\,2) + q^{10n+11}(\bar{1})$
9_{42}	4	FA, $\{1\}su(n)_q$	$1^3\bar{2}3/\bar{1}^2\bar{2}3$	$(0\,\bar{1})$	$q^{-2n-11}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1}) + q^{-11}(1\,0\,0\,1\,1\,0\,0\,0\,0\,1) + q^{2n-5}(\bar{1}\,\bar{1}\,0\,1\,\bar{2}\,\bar{1}) + q^{4n+3}(1\,0\,\bar{1})$
9_{43}	4		$12^3 3/\bar{2}3^2/1\overline{23}$	$q^{2n}(1)+q^{4n}(\bar{1}1\bar{1})+q^{6n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-11}(\bar{1}00\bar{1}00\bar{1}0\bar{1}\bar{1}00\bar{1})+q^{8n-11}(1\bar{1}\bar{1}31\bar{1}2201102 0\bar{1}1)+q^{10n-5}(\bar{1}\,0\,1\,\bar{1}\,\bar{2}\,1\,1\,\bar{3}\,0\,0\,\bar{1}\,0\,\bar{1}) + q^{12n+3}(1\,0\,0\,1\,0\,0\,1\,1) + q^{14n+13}(\bar{1})$
9_{44}	4	10_{71}, $\{2\}su(n)_q$	$\bar{1}23/\overline{12}3/2^2\bar{3}$	$(\bar{1}) + q^{2n}(\bar{1}\,1)$	$q^{-2n-3}(\bar{1}) + q^{-7}(1\,1\,0\,\bar{1}\,0\,1) + q^{2n-7}(\bar{1}\,\bar{1}\,3\,\bar{2}\,\bar{4}\,2\,1\,\bar{2}) + q^{4n-5}(1\,\bar{1}\,1\,3\,\bar{3}\,0\,4\,\bar{1}\,\bar{1}\,1) + q^{6n+1}(\bar{1}\,1\,0\,\bar{2}\,1\,1\,\bar{1})$
9_{45}	4		$1\bar{2}3/12^3 3/\bar{2}$	$q^{2n}(1) + q^{4n}(2\,\bar{1}) + q^{6n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-5}(\bar{1}\bar{1}40\bar{5}22\bar{2})+q^{8n-1}(\bar{1}43\bar{7}16\bar{2}\bar{1}1)+q^{10n+1}(\bar{1}13\bar{5}\bar{3}40\bar{2})+q^{12n+7}(2\,\bar{1}\,\bar{2}\,2\,1) + q^{14n+13}(\bar{1})$
9_{46}	4		$12\bar{3}/\bar{1}23/1\bar{2}3$	$q^{2n}(\bar{1}) + q^{4n}(\bar{1})$	$q^{2n-3}(\bar{1}\,\bar{1}\,\bar{1}) + q^{4n-3}(1\,\bar{1}\,\bar{2}) + q^{6n+1}(1\,0\,\bar{1}\,1\,1) + q^{8n+5}(1\,\bar{1}\,\bar{1}\,1) + q^{10n+11}(\bar{1})$
9_{47}	4		$\bar{1}23/\bar{1}23/\bar{1}23$	$q^{2n}(2\,\bar{2}\,1) + q^{4n}(\bar{1})$	$q^{2n-13}(\bar{1}22\bar{4}03\bar{2}\bar{2}\bar{2}02\bar{1})+q^{4n-13}(1\bar{2}\bar{1}6\bar{3}\bar{8}64\bar{9}12\bar{4}40\bar{2}1)+q^{6n-7}(\bar{1}13\bar{5}\bar{3}10\bar{3}\bar{7}50\bar{1}1\bar{1})+q^{8n+1}(2\,\bar{3}\,0\,5\,\bar{3}\,\bar{2}\,2\,1) + q^{10n+7}(\bar{1}\,1\,\bar{1}\,\bar{1})$
9_{48}	4		$1^2\bar{2}^2 3^2/2/\bar{1}2\bar{3}/2$	$q^{2n}(3\,\bar{1}) + q^{4n}(2)$	$q^{2n-7}(\bar{1}\,2\,1\,\bar{4}\,2\,2\,\bar{1}) + q^{4n-7}(1\,\bar{3}\,0\,9\,\bar{6}\,\bar{6}\,9\,0\,\bar{3}\,1) + q^{6n-3}(\bar{2}\,0\,9\,\bar{2}\,\bar{9}\,6\,4\,\bar{3}) + q^{8n+3}(5\,0\,\bar{6}\,2\,3) + q^{10n+9}(\bar{3}\,\bar{1})$
9_{49}	4		$1^2 23^2/\bar{1}2^2 3/\bar{2}3$	$q^{2n}(1)+q^{4n}(0\,1)+q^{6n}(\bar{1}\,2)$	$q^{2n+1}(1) + q^{4n-1}(1\,0\,0\,1) + q^{6n-3}(1\,0\,0\,1\,0\,0\,1) + q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1) + q^{10n-5}(2\,1\,\bar{4}\,2\,5\,\bar{4}\,\bar{1}\,4\,\bar{2}\,0\,2) + q^{12n-3}(1\,3\,\bar{4}\,\bar{2}\,10\,\bar{2}\,\bar{7}\,7\,0\,\bar{3}\,3)$
10_1	6		$\overline{12}3\bar{5}/2\bar{4}/1^2 234\bar{5}/4$	$(\bar{1}) + q^{2n}(\bar{1}) + q^{4n}(\bar{1}) + q^{6n}(\bar{1})$	$q^{-2n-3}(\bar{1})+q^{-3}(1\bar{2}\bar{1}1)+q^{2n-3}(\bar{1}20\bar{2}1)+q^{4n-7}(\bar{1}00\bar{3}23\bar{4}\bar{1}1)+q^{6n-7}(2\bar{2}\bar{2}4\bar{2}15\bar{5}\bar{2}3)+q^{8n-7}(\bar{1}\,2\,2\,\bar{8}\,2\,8\,\bar{6}\,0\,3\,\bar{5}\,1\,2\,\bar{1}) + q^{10n-1}(1\,\bar{1}\,\bar{2}\,2\,2\,\bar{2}\,0\,0\,\bar{1}\,1) + q^{12n+5}(1\,\bar{2}\,0\,4\,\bar{2}\,\bar{1}\,1) + q^{14n+11}(\bar{1}\,1\,0\,\bar{1})$
10_2	3		$1\bar{2}/1^7\bar{2}$	$q^{2n}(1) + q^{4n}(0\ \ 1) + q^{6n}(1\,\bar{1}\,1\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-15}(\bar{1}\,0\,0\,\bar{1}\,0\,0\,\bar{1}\,0\,\bar{1}\,\bar{1}\,0\,\bar{1}\,\bar{2}\,0\,\bar{1}\,\bar{1}\,0\,0\,\bar{1})+q^{12n-15}(1\,\bar{1}\,\bar{1}\,3\,0\,\bar{2}\,2\,1\,\bar{1}\,1\,0\,0\,2\,\bar{1}\,\bar{1}\,2\,0\,\bar{1}\,2\,0\,\bar{1}\,1)+q^{14n-9}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1\,0\,0\,0\,\bar{1}\,0\,1\,\bar{1}\,0\,1\,\bar{1})$
10_3	6		$1^2 23\bar{4}5/3\bar{4}/\bar{1}23\bar{5}^2$	$q^{-2n}(\bar{1}) + (\bar{2}) + q^{2n}(\bar{2}) + q^{4n}(\bar{1})$	$q^{-6n-7}(\bar{1})+q^{-4n-5}(\bar{2}\,\bar{1}\,1)+q^{-2n-3}(\bar{3}\,\bar{2}\,1\,1)+q^{-7}(1\,1\,0\,\bar{5}\,\bar{3}\,3\,1)+q^{2n-5}(1\,3\,\bar{1}\,\bar{6}\,\bar{1}\,3\,1)+q^{4n-3}(1\,1\,1\,\bar{4}\,\bar{3}\,3\,1)+q^{6n+1}(1\,1\,\bar{2}\,\bar{3}\,1\,1)+q^{8n+5}(1\,\bar{1}\,\bar{2})+q^{10n+11}(\bar{1})$
10_4	5		$1^2 23\bar{4}/\bar{2}3/\bar{1}2\bar{4}^3$	$q^{-2n}(1\,\bar{1})+(0\,\bar{1})+q^{2n}(0\,\bar{1})$	$q^{-6n-11}(\bar{1}\,1\,0\,\bar{2}\,1\,1\,\bar{1}) + q^{-4n-9}(\bar{1}\,0\,0\,\bar{2}\,1\,2\,\bar{2}\,\bar{1}\,1) + q^{-2n-7}(\bar{1}\,\bar{1}\,\bar{1}\,0\,1\,0\,\bar{2}\,0\,1) + q^{-11}(1\,0\,0\,0\,\bar{1}\,\bar{1}\,0\,1\,0\,\bar{1}\,0\,1) + q^{2n-7}(1\,0\,\bar{1}\,1\,\bar{1}\,\bar{2}\,2\,1\,\bar{2}\,0\,1) + q^{4n-3}(1\,\bar{1}\,0\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1) + q^{6n+3}(\bar{1}\,0\,\bar{1}\,\bar{1}\,0\,0\,\bar{1})$
10_5	3		$\bar{1}^2 2/\bar{1}2^6$	$q^{2n}(1) + q^{4n}(\bar{1}\,2\,\bar{1}\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-17}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,0\,1\,0\,1\,0\,\bar{1}\,1\,1\,\bar{1}\,0\,1\,\bar{1})+q^{8n-17}(1\,\bar{1}\,0\,4\,\bar{2}\,\bar{3}\,6\,1\,\bar{4}\,3\,2\,\bar{1}\,2\,0\,\bar{1}\,4\,0\,\bar{2}\,3\,0\,\bar{1}\,1)+q^{10n-11}(\bar{1}\,1\,0\,\bar{3}\,1\,2\,\bar{4}\,\bar{1}\,2\,\bar{2}\,\bar{2}\,1\,\bar{2}\,0\,1\,\bar{2}\,0\,1\,\bar{1})$
10_6	4		$1\bar{2}3/\overline{12}3/\bar{1}2^6$	$q^{2n}(1) + q^{4n}(\bar{1}\ \ 1\ \ \bar{1}) + q^{6n}(\bar{1}\,1\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-11}(\bar{1}00\bar{1}00\bar{1}0\bar{1}\bar{1}00\bar{1})+q^{8n-9}(\bar{2}\,0\,3\,\bar{2}\,\bar{2}\,3\,0\,\bar{2}\,1\,\bar{1}\,\bar{1}\,2\,\bar{1}\,\bar{1}\,1)+q^{10n-11}(1\,0\,\bar{3}\,2\,5\,\bar{4}\,\bar{3}\,6\,1\,\bar{4}\,2\,0\,\bar{1}\,3\,\bar{1}\,\bar{1}\,1)+q^{12n-7}(1\,\bar{2}\,\bar{1}\,5\,\bar{2}\,\bar{5}\,5\,2\,\bar{5}\,2\,1\,\bar{3}\,3\,0\,\bar{2}\,1)+q^{14n-1}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1\,\bar{1}\,0\,1\,\bar{1})$
10_7	5	8_{11}, $\{2\}su(n)_q$	$1^2 23\bar{4}/\bar{1}23\bar{4}/3$	$q^{2n}(2\bar{1})+q^{4n}(2\bar{1})+q^{6n}(1\bar{1})$	$q^{2n-7}(\bar{1}11\bar{2}11\bar{1})+q^{4n-5}(\bar{3}06\bar{4}\bar{4}5\bar{1}\bar{2}1)+q^{6n-7}(10\bar{4}17\bar{4}\bar{6}50\bar{2}1)+q^{8n-3}(1\bar{2}\bar{1}6\bar{1}\bar{5}41\bar{2}1)+q^{10n+3}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_8	4		$1^5\overline{23}^2/1\overline{23}$	$q^{2n}(0\,0\,\bar{1})+q^{4n}(\bar{1}\,1\,\bar{1})$	$q^{2n-15}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1}\,\bar{1}\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})+q^{4n-15}(1\,\bar{1}\,\bar{1}\,2\,0\,\bar{2}\,1\,2\,\bar{1}\,0\,1\,\bar{1}\,1\,0\,\bar{1}\,1)+q^{6n-13}(1\,0\,\bar{2}\,1\,2\,\bar{1}\,\bar{1}\,2\,1\,\bar{1}\,0\,0\,0\,1\,\bar{1}\,0\,1)+q^{8n-11}(1\,\bar{1}\,\bar{2}\,2\,1\,\bar{2}\,0\,1\,0\,0\,\bar{1}\,\bar{1}\,1\,0\,\bar{1})+q^{10n-5}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1\,\bar{1}\,0\,1\,\bar{1})$
10_9	3		$\bar{1}2^5/\bar{1}^3 2$	$q^{2n}(2\,\bar{2}\,1\,\bar{1})$	$q^{2n-19}(\bar{1}\,1\,0\,\bar{3}\,1\,2\,\bar{4}\,\bar{1}\,2\,\bar{2}\,\bar{1}\,1\,\bar{2}\,0\,1\,\bar{2}\,0\,1\,\bar{1})+q^{4n-19}(1\,\bar{1}\,0\,3\,\bar{3}\,\bar{1}\,7\,\bar{3}\,\bar{5}\,8\,1\,\bar{6}\,5\,1\,\bar{3}\,4\,\bar{1}\,\bar{2}\,3\,0\,\bar{1}\,1)+q^{6n-13}(\bar{1}\,1\,0\,\bar{2}\,2\,1\,\bar{4}\,2\,3\,\bar{5}\,0\,3\,\bar{3}\,0\,1\,\bar{2}\,0\,1\,\bar{1})$
10_{10}	5		$\bar{1}2\bar{3}\bar{4}/1^2 2\bar{3}/2\bar{3}^2 4/3^2$	$(\bar{2}\,1)+q^{2n}(\bar{2}\,1)+q^{4n}(\bar{1}\,1)$	$q^{-2n-9}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})+q^{-7}(\bar{1}\,3\,2\,\bar{8}\,0\,7\,\bar{2}\,\bar{2}\,1)+q^{2n-7}(\bar{1}\,0\,5\,\bar{1}\,\bar{9}\,3\,7\,\bar{3}\,\bar{2}\,1)+q^{4n-9}(1\,0\,\bar{2}\,1\,5\,\bar{1}\,\bar{8}\,3\,7\,\bar{3}\,\bar{2}\,1)+q^{6n-5}(1\,\bar{1}\,\bar{3}\,5\,2\,\bar{9}\,1\,7\,\bar{3}\,\bar{2}\,1)+q^{8n-1}(1\,\bar{2}\,0\,5\,\bar{4}\,\bar{3}\,6\,0\,\bar{2}\,1)+q^{10n+5}(\bar{1}\,1\,0\,\bar{2}\,1\,1\,\bar{1})$
10_{11}	5		$\bar{1}^2\bar{2}3^3\bar{4}/1\bar{2}34^2$	$(0\,\bar{1})+q^{2n}(2\,\bar{2})+q^{4n}(1\,\bar{1})$	$q^{-2n-11}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})+q^{-9}(\bar{2}\,0\,2\,\bar{2}\,\bar{2}\,1\,\bar{1}\,\bar{1}\,1)+q^{2n-11}(1\,1\,\bar{4}\,0\,6\,\bar{3}\,\bar{3}\,3\,\bar{1}\,\bar{2}\,1\,1)+q^{4n-9}(1\,3\,\bar{6}\,\bar{4}\,13\,\bar{1}\,\overline{11}\,7\,2\,\bar{5}\,2\,1)+q^{6n-7}(1\,1\,\bar{4}\,\bar{3}\,9\,1\,\overline{10}\,3\,4\,\bar{4}\,0\,1)+q^{8n-3}(1\,\bar{2}\,\bar{2}\,5\,0\,\bar{5}\,2\,1\,\bar{2})+q^{10n+3}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
10_{12}	4		$\bar{1}^2 23^2/\bar{1}2^4\bar{3}$	$q^{2n}(2\,\bar{1}\,1)+q^{4n}(2\,\bar{1}\,1)$	$q^{2n-13}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,0\,1\,\bar{1}\,0\,1\,\bar{1})+q^{4n-11}(\bar{1}\,2\,2\,\bar{5}\,0\,7\,\bar{3}\,\bar{3}\,5\,\bar{1}\,\bar{1}\,3\,\bar{1}\,\bar{1}\,1)+q^{6n-13}(1\,0\,\bar{2}\,4\,3\,\bar{8}\,2\,11\,\bar{7}\,\bar{4}\,9\,\bar{2}\,\bar{2}\,4\,\bar{1}\,\bar{1}\,1)+q^{8n-9}(1\,\bar{2}\,0\,5\,\bar{5}\,\bar{4}\,10\,\bar{3}\,\bar{8}\,8\,0\,\bar{5}\,4\,0\,\bar{2}\,1)+q^{10n-3}(\bar{1}\,1\,0\,\bar{3}\,1\,2\,\bar{4}\,0\,2\,\bar{2}\,0\,1\,\bar{1})$
10_{14}	4		$\bar{1}23^2/\bar{1}2^4\bar{3}/2$	$q^{2n}(1)+q^{4n}(\bar{1}\,2\,\bar{1})+q^{6n}(\bar{2}\,2\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-11}(\bar{1}\,1\,1\,\bar{2}\,1\,2\,\bar{1}\,0\,1\,\bar{1}\,1\,1\,\bar{1})+q^{8n-9}(\bar{3}\,1\,6\,\bar{6}\,\bar{4}\,9\,\bar{1}\,\bar{6}\,5\,\bar{1}\,\bar{3}\,5\,\bar{1}\,\bar{2}\,1)+q^{10n-11}(1\,0\,\bar{5}\,3\,10\,\overline{10}\,\bar{8}\,16\,\bar{1}\,\overline{12}\,8\,0\,\bar{7}\,6\,0\,\bar{3}\,1)+q^{12n-7}(1\,\bar{3}\,\bar{1}\,10\,\bar{5}\,\overline{12}\,14\,4\,\overline{15}\,7\,4\,\bar{9}\,5\,2\,\bar{3}\,1)+q^{14n-1}(\bar{1}\,2\,1\,\bar{5}\,2\,5\,\bar{5}\,\bar{1}\,4\,\bar{3}\,0\,2\,\bar{1})$
10_{13}	6		$1^2 2\bar{3}45^2/4/3\bar{4}\bar{5}/\bar{1}\bar{2}^2\bar{3}$	$q^{-2n}(\bar{1})+(\bar{4}\,1)+q^{2n}(\bar{3}\,1)+q^{4n}(\bar{1})$	$q^{-10n-11}(\bar{1})+q^{-8n-13}(1\,1\,\bar{3}\,\bar{1}\,2)+q^{-6n-13}(\bar{2}\,3\,5\,\bar{8}\,\bar{5}\,5\,1\,\bar{1})+q^{-4n-13}(1\,\bar{4}\,1\,14\,\bar{9}\,\overline{15}\,10\,5\,\bar{3})+q^{-2n-11}(1\,\bar{4}\,\bar{1}\,16\,\bar{4}\,\overline{22}\,6\,12\,\bar{4}\,\bar{2}\,1)+q^{-9}(1\,\bar{4}\,\bar{1}\,16\,\bar{5}\,\overline{20}\,6\,8\,\bar{2}\,\bar{1})+q^{2n-5}(\bar{1}\,0\,7\,\bar{1}\,\overline{12}\,2\,5\,\bar{1})+q^{4n+1}(2\,\bar{1}\,\bar{4}\,0\,1)+q^{6n+7}(\bar{1})$
10_{15}	4		$\bar{1}\bar{2}3^4/\bar{2}/\bar{1}^2 23$	$(1\,\bar{1}\,1)+q^{2n}(2\,\bar{1}\,1)$	$q^{-2n-17}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1\,\bar{1}\,0\,1\,\bar{1})+q^{-17}(1\,\bar{2}\,\bar{1}\,5\,\bar{2}\,\bar{5}\,5\,2\,\bar{5}\,2\,1\,\bar{3}\,3\,0\,\bar{2}\,1)+q^{2n-15}(1\,\bar{1}\,\bar{2}\,5\,2\,\bar{7}\,2\,7\,\bar{3}\,\bar{2}\,4\,\bar{2}\,1\,3\,\bar{2}\,0\,1)+q^{4n-13}(1\,\bar{1}\,\bar{1}\,4\,0\,\bar{5}\,3\,4\,\bar{3}\,0\,2\,\bar{1}\,1\,1\,\bar{1})+q^{6n-7}(\bar{1}\,1\,0\,\bar{3}\,1\,2\,\bar{4}\,0\,2\,\bar{2}\,0\,1\,\bar{1})$
10_{16}	5		$1\bar{2}\bar{3}\bar{4}/\bar{1}^2\bar{2}34^2/3^2$	$(1\,\bar{1})+q^{2n}(2\,\bar{2})+q^{4n}(1\,\bar{1})$	$q^{-2n-11}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})+q^{-9}(\bar{3}\,0\,5\,\bar{4}\,\bar{4}\,4\,\bar{1}\,\bar{2}\,1)+q^{2n-11}(1\,1\,\bar{5}\,\bar{2}\,8\,\bar{1}\,\bar{8}\,3\,2\,\bar{3}\,0\,1)+q^{4n-9}(1\,3\,\bar{6}\,\bar{5}\,13\,2\,\overline{12}\,4\,4\,\bar{4}\,1\,1)+q^{6n-7}(1\,1\,\bar{4}\,\bar{3}\,9\,2\,\bar{9}\,2\,4\,\bar{3}\,0\,1)+q^{8n-3}(1\,\bar{2}\,\bar{2}\,5\,0\,\bar{5}\,2\,1\,\bar{2})+q^{10n+3}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
10_{17}	3	A	$1^4\bar{2}/1\bar{2}^4$	$(\bar{2}\,2\,\bar{1}\,1)$	$q^{-2n-21}(\bar{1}\,1\,0\,\bar{2}\,2\,1\,\bar{4}\,2\,3\,\bar{5}\,0\,3\,\bar{3}\,0\,1\,\bar{2}\,0\,1\,\bar{1})+q^{-21}(1\,\bar{1}\,0\,3\,\bar{2}\,0\,5\,\bar{4}\,\bar{1}\,10\,\bar{5}\,\bar{5}\,10\,\bar{1}\,\bar{4}\,5\,0\,\bar{2}\,3\,0\,\bar{1}\,1)+q^{2n-15}(\bar{1}\,1\,0\,\bar{2}\,1\,0\,\bar{3}\,3\,0\,\bar{5}\,3\,2\,\bar{4}\,1\,2\,\bar{2}\,0\,1\,\bar{1})$
10_{18}	5		$\bar{1}^2\bar{2}3^2 4^2/1\bar{3}4/\bar{2}3$	$(1\,\bar{1})+q^{2n}(3\,\bar{2})+q^{4n}(2\,\bar{1})$	$q^{-2n-11}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})+q^{-9}(\bar{3}\,1\,5\,\bar{5}\,\bar{3}\,5\,\bar{1}\,\bar{2}\,1)+q^{2n-11}(1\,1\,\bar{6}\,0\,11\,\bar{5}\,\bar{8}\,7\,2\,\bar{4}\,0\,1)+q^{4n-9}(1\,3\,\bar{8}\,\bar{5}\,19\,\bar{2}\,\overline{18}\,10\,6\,\bar{7}\,0\,1)+q^{6n-7}(1\,1\,\bar{6}\,\bar{3}\,14\,0\,\overline{15}\,5\,7\,\bar{5}\,\bar{1}\,1)+q^{8n-3}(1\,\bar{3}\,\bar{2}\,9\,\bar{1}\,\bar{9}\,4\,3\,\bar{2})+q^{10n+3}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})$
10_{19}	4		$1\bar{2}3/\bar{2}/\bar{1}^2\bar{2}3^4$	$(1\,\bar{1}\,1)+q^{2n}(2\,\bar{2}\,1)$	$q^{-2n-17}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1\,\bar{1}\,0\,1\,\bar{1})+q^{-17}(1\,\bar{2}\,0\,6\,\bar{3}\,\bar{5}\,7\,2\,\bar{6}\,3\,1\,\bar{3}\,4\,0\,\bar{2}\,1)+q^{2n-15}(1\,\bar{2}\,\bar{2}\,7\,\bar{1}\,\overline{10}\,5\,7\,\bar{8}\,\bar{1}\,4\,\bar{4}\,2\,3\,\bar{3}\,0\,1)+q^{4n-13}(1\,\bar{2}\,\bar{2}\,7\,\bar{1}\,\overline{10}\,6\,7\,\bar{8}\,0\,4\,\bar{4}\,2\,2\,\bar{2})+q^{6n-7}(\bar{1}\,2\,1\,\bar{5}\,2\,5\,\bar{5}\,\bar{1}\,4\,\bar{3}\,0\,2\,\bar{1})$
10_{20}	5		$\bar{1}2^3 34^2/3\bar{4}/\bar{1}2\bar{3}$	$q^{2n}(1\,\bar{1})+q^{4n}(1\,\bar{1})+q^{6n}(1\,\bar{1})$	$q^{2n-7}(\bar{1}\,0\,0\,\bar{1}\,0\,0\,\bar{1})+q^{4n-5}(\bar{2}\,0\,3\,\bar{2}\,\bar{2}\,2\,\bar{1}\,\bar{1}\,1)+q^{6n-5}(\bar{1}\,\bar{2}\,2\,3\,\bar{4}\,\bar{2}\,3\,\bar{1}\,\bar{1}\,1)+q^{8n-7}(1\,0\,\bar{2}\,0\,4\,1\,\bar{4}\,0\,3\,\bar{1}\,\bar{1}\,1)+q^{10n-3}(1\,\bar{1}\,\bar{3}\,4\,3\,\bar{6}\,0\,4\,\bar{2}\,\bar{1}\,1)+q^{12n+1}(1\,\bar{2}\,\bar{1}\,5\,\bar{2}\,\bar{4}\,4\,0\,\bar{2}\,1)+q^{14n+7}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
10_{21}	4		$\bar{1}23^2/\bar{1}2\bar{3}/2^4$	$q^{2n}(1)+q^{4n}(\bar{1}\,2\,\bar{1})+q^{6n}(\bar{1}\,1\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-11}(\bar{1}\,1\,1\,\bar{2}\,1\,2\,\bar{1}\,0\,1\,\bar{1}\,1\,1\,\bar{1})+q^{8n-9}(\bar{3}\,0\,6\,\bar{4}\,\bar{5}\,7\,1\,\bar{6}\,3\,0\,\bar{3}\,4\,\bar{1}\,\bar{2}\,1)+q^{10n-11}(1\,0\,\bar{4}\,1\,7\,\bar{4}\,\bar{8}\,7\,4\,\bar{8}\,0\,2\,\bar{4}\,3\,0\,\bar{2}\,1)+q^{12n-7}(1\,\bar{2}\,\bar{1}\,6\,\bar{1}\,\bar{6}\,5\,4\,\bar{5}\,1\,2\,\bar{3}\,3\,1\,\bar{2}\,1)+q^{14n-1}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1\,\bar{1}\,0\,1\,\bar{1})$
10_{22}	4	10_{35}, $\{1\}su(2)_q$	$\bar{1}^3 23^2/\bar{1}2^3\bar{3}$	$(\bar{2}\,1\,\bar{1})+q^{2n}(\bar{2}\,1\,\bar{1})$	$q^{-2n-15}(\bar{1}\,1\,0\,\bar{3}\,1\,2\,\bar{4}\,0\,2\,\bar{2}\,0\,1\,\bar{1})+q^{-13}(\bar{1}\,1\,0\,\bar{4}\,3\,3\,\bar{8}\,0\,6\,\bar{4}\,\bar{2}\,3\,\bar{1}\,\bar{1}\,1)+q^{2n-15}(1\,0\,\bar{2}\,3\,1\,\bar{6}\,7\,4\,\overline{12}\,4\,9\,\bar{7}\,\bar{2}\,5\,\bar{1}\,\bar{1}\,1)+q^{4n-11}(1\,\bar{2}\,0\,4\,\bar{5}\,0\,8\,\bar{8}\,\bar{3}\,10\,\bar{4}\,\bar{5}\,5\,0\,\bar{2}\,1)+q^{6n-5}(\bar{1}\,1\,0\,\bar{2}\,2\,0\,\bar{4}\,2\,1\,\bar{3}\,0\,1\,\bar{1})$
10_{23}	4		$23^2/\bar{1}^2 2/\bar{1}2^3\bar{3}$	$q^{2n}(3\,\bar{2}\,1)+q^{4n}(2\,\bar{1}\,1)$	$q^{2n-13}(\bar{1}\,2\,1\,\bar{5}\,2\,5\,\bar{5}\,0\,4\,\bar{3}\,0\,2\,\bar{1})+q^{4n-11}(\bar{2}\,3\,5\,\overline{11}\,\bar{2}\,18\,\bar{8}\,\overline{11}\,14\,\bar{2}\,\bar{6}\,6\,\bar{1}\,\bar{2}\,1)+q^{6n-13}(1\,0\,\bar{3}\,4\,6\,\overline{11}\,\bar{4}\,19\,\bar{5}\,\overline{15}\,13\,2\,\bar{8}\,5\,0\,\bar{2}\,1)+q^{8n-9}(1\,\bar{2}\,0\,6\,\bar{5}\,\bar{6}\,12\,0\,\overline{11}\,8\,3\,\bar{6}\,4\,1\,\bar{2}\,1)+q^{10n-3}(\bar{1}\,1\,0\,\bar{3}\,1\,2\,\bar{4}\,0\,2\,\bar{2}\,0\,1\,\bar{1})$
10_{24}	5		$\bar{1}2\bar{3}/2^2/1^2 23\bar{4}/3\bar{4}$	$q^{2n}(2\,\bar{1})+q^{4n}(3\,\bar{2})+q^{6n}(1\,\bar{1})$	$q^{2n-7}(\bar{1}\,1\,1\,\bar{2}\,1\,1\,\bar{1})+q^{4n-7}(\bar{1}\,\bar{3}\,3\,5\,\bar{7}\,\bar{2}\,6\,\bar{2}\,\bar{2}\,1)+q^{6n-7}(1\,\bar{3}\,\bar{7}\,11\,8\,\overline{18}\,\bar{1}\,12\,\bar{5}\,\bar{3}\,2)+q^{8n-7}(1\,1\,\bar{3}\,\bar{7}\,10\,12\,\overline{19}\,\bar{5}\,16\,\bar{4}\,\bar{5}\,3)+q^{10n-3}(2\,\bar{1}\,\bar{7}\,6\,12\,\overline{12}\,\bar{7}\,12\,\bar{1}\,\bar{4}\,2)+q^{12n+1}(1\,\bar{3}\,\bar{1}\,8\,\bar{3}\,\bar{7}\,6\,1\,\bar{3}\,1)+q^{14n+7}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
10_{25}	4	10_{56}, $\{1\}su(n)_q$	$\bar{1}23^2/\bar{1}2^3\bar{3}/2^2$	$q^{2n}(1)+q^{4n}(\bar{2}\,2\,\bar{1})+q^{6n}(\bar{3}\,2\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-11}(\bar{1}\,1\,1\,\bar{3}\,1\,3\,\bar{3}\,0\,2\,\bar{2}\,0\,1\,\bar{1})+q^{8n-9}(\bar{3}\,1\,7\,\bar{8}\,\bar{6}\,13\,\bar{4}\,\overline{10}\,10\,\bar{1}\,\bar{7}\,5\,0\,\bar{2}\,1)+q^{10n-11}(1\,0\,\bar{5}\,3\,12\,\overline{12}\,\overline{11}\,22\,\bar{2}\,\overline{18}\,14\,3\,\overline{12}\,6\,2\,\bar{3}\,1)+q^{12n-7}(1\,\bar{3}\,\bar{1}\,11\,\bar{6}\,\overline{15}\,19\,5\,\overline{21}\,11\,8\,\overline{12}\,4\,3\,\bar{3}\,1)+q^{14n-1}(\bar{1}\,2\,1\,\bar{6}\,3\,6\,\bar{8}\,0\,6\,\bar{4}\,\bar{1}\,2\,\bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_{26}	4		$12^2\bar3/\bar2^3/1\bar23/1^2\bar2$	$(\bar3\ 2\ \bar1) + q^{2n}(\bar2\ 1\ \bar1)$	$q^{-2n-15}(\bar1\ 2\ 0\ \bar5\ 5\ 3\ \bar9\ 3\ 5\ \bar5\ 0\ 2\ \bar1) + q^{-13}(\bar2\ 2\ 2\ \overline{10}\ 6\ 11\ \overline{19}\ \bar1\ 17\ \bar9\ \bar6\ 7\ \bar1\ \bar2\ 1) + q^{2n-15}(1\ 0\ \bar3\ 3\ 3\ \overline{10}\ 5\ 12\ \overline{18}\ \bar3\ 18\ \bar7\ \bar8\ 7\ 0\ \bar2\ 1) + q^{4n-11}(1\ \bar2\ 0\ 5\ \bar5\ \bar1\ 11\ \bar8\ \bar6\ 13\ \bar2\ \bar7\ 5\ 1\ \bar2\ 1) + q^{6n-5}(\bar1\ 1\ 0\ \bar2\ 2\ 0\ \bar4\ 2\ 1\ \bar3\ 0\ 1\ \bar1)$
10_{27}	4		$23^4/\bar12/\bar1^22\bar3$	$q^{2n}(3\ \bar2\ 1) + q^{4n}(3\ \bar2\ 1)$	$q^{2n-13}(\bar1\ 2\ 1\ \bar5\ 2\ 5\ \bar5\ 0\ 4\ \bar3\ 0\ 2\ \bar1) + q^{4n-11}(\bar2\ 4\ 4\ \overline{13}\ 2\ 18\ \overline{13}\ \bar8\ 16\ \bar5\ \bar6\ 7\ \bar1\ \bar2\ 1) + q^{6n-13}(1\ 0\ \bar4\ 6\ 7\ \overline{19}\ 2\ 27\ \overline{19}\ \overline{14}\ 24\ \bar5\ \overline{11}\ 9\ 0\ \bar3\ 1) + q^{8n-9}(1\ \bar3\ 0\ 9\ \overline{11}\ \bar8\ 24\ \bar8\ \overline{20}\ 20\ 2\ \overline{13}\ 7\ 2\ \bar3\ 1) + q^{10n-3}(\bar1\ 2\ 0\ \bar5\ 5\ 3\ \bar9\ 3\ 5\ \bar5\ 0\ 2\ \bar1)$
10_{28}	5		$1^22\overline{34}/23\bar4^2/\bar13/2$	$(\bar2\ 1) + q^{2n}(\bar2\ 2) + q^{4n}(\bar1\ 1)$	$q^{-2n-9}(\bar1\ 2\ 1\ \bar4\ 1\ 2\ \bar1) + q^{-9}(\bar1\ \bar1\ 5\ 1\ \bar9\ 2\ 7\ \bar3\ \bar2\ 1) + q^{2n-9}(1\ \bar2\ \bar2\ 9\ \bar2\ \overline{13}\ 7\ 9\ \bar7\ \bar2\ 2) + q^{4n-9}(1\ 1\ \bar2\ \bar1\ 8\ 0\ \overline{12}\ 6\ 11\ \bar6\ \bar3\ 3) + q^{6n-5}(2\ \bar1\ \bar4\ 7\ 3\ \overline{12}\ 3\ 10\ \bar4\ \bar3\ 2) + q^{8n-1}(1\ \bar3\ 0\ 5\ \bar6\ \bar3\ 7\ \bar1\ \bar3\ 1) + q^{10n+5}(\bar1\ 1\ 0\ \bar2\ 1\ 1\ \bar1)$
10_{29}	5		$1\bar23\bar4/3/\overline{12}34/\overline{12}3^3$	$(0\ \bar1) + q^{2n}(3\ \bar3\ 1) + q^{4n}(1\ \bar1)$	$q^{-2n-11}(\bar1\ 0\ 0\ \bar1\ \bar1\ 0\ \bar1) + q^{-13}(1\ 1\ \bar3\ 0\ 4\ \bar2\ \bar3\ 2\ \bar1\ \bar1\ 2) + q^{2n-13}(\bar2\ 3\ 4\ \overline{10}\ \bar2\ 13\ \bar5\ \bar8\ 7\ \bar2\ \bar4\ 4\ 0\ \bar1) + q^{4n-13}(1\ \bar3\ 1\ 11\ \overline{12}\ \overline{12}\ 25\ 0\ \overline{22}\ 14\ 4\ \overline{12}\ 7\ 2\ \bar3\ 1) + q^{6n-9}(\bar1\ 0\ 6\ \bar5\ \overline{11}\ 15\ 6\ \overline{19}\ 5\ 9\ \overline{10}\ 1\ 4\ \bar2) + q^{8n-3}(2\ \bar2\ \bar4\ 6\ 2\ \bar7\ 2\ 3\ \bar3\ 0\ 1) + q^{10n+3}(\bar1\ 1\ 1\ \bar2\ 0\ 1\ \bar1)$
10_{30}	5		$1^22^23\bar4/\bar234/\bar12\overline{34}/2$	$q^{2n}(3\bar1)+q^{4n}(4\bar2)+q^{6n}(2\bar1)$	$q^{2n-7}(\bar1\ 2\ 1\ \bar4\ 2\ 2\ \bar1) + q^{4n-7}(\bar1\ \bar3\ 6\ 7\ \overline{12}\ \bar2\ 11\ \bar2\ \bar3\ 1) + q^{6n-7}(1\ \bar4\ \bar8\ 16\ 11\ \overline{26}\ \bar2\ 19\ \bar5\ \bar5\ 2) + q^{8n-7}(1\ 1\ \bar5\ \bar8\ 16\ 15\ \overline{28}\ \bar7\ 23\ \bar4\ \bar7\ 3) + q^{10n-3}(2\ \bar2\ \bar9\ 11\ 16\ \overline{21}\ \overline{10}\ 18\ \bar1\ \bar6\ 2) + q^{12n+1}(1\ \bar4\ 0\ 12\ \bar7\ \overline{11}\ 10\ 2\ \bar4\ 1) + q^{14n+7}(\bar1\ 2\ 1\ \bar4\ 1\ 2\ \bar1)$
10_{31}	5		$1^22^2\overline{34}^2/\overline{13}/2\bar34$	$q^{-2n}(\bar1\ 1) + (\bar3\ 2) + q^{2n}(\bar2\ 1)$	$q^{-6n-13}(\bar1\ 1\ 1\ \bar2\ 0\ 1\ \bar1) + q^{-4n-11}(\bar2\ 2\ 4\ \bar5\ \bar2\ 5\ \bar1\ \bar2\ 1) + q^{-2n-13}(1\ 0\ \bar5\ 6\ 9\ \overline{12}\ \bar4\ 11\ 0\ \bar4\ 1\ 1) + q^{-11}(1\ 1\ \bar8\ 5\ 17\ \overline{18}\ \overline{12}\ 20\ 1\ \bar8\ 2\ 1) + q^{2n-9}(1\ 0\ \bar6\ 3\ 13\ \overline{12}\ \overline{12}\ 14\ 4\ \bar7\ 0\ 1) + q^{4n-5}(1\ \bar3\ \bar1\ 9\ \bar4\ \bar9\ 7\ 3\ \bar3) + q^{6n+1}(\bar1\ 2\ 1\ \bar4\ 1\ 2\ \bar1)$
10_{32}	4		$1^22^2/\bar12\bar3/2\bar3^3$	$(\bar3\ 2\ \bar1) + q^{2n}(\bar2\ 2\ \bar1)$	$q^{-2n-15}(\bar1\ 2\ 0\ \bar5\ 5\ 3\ \bar9\ 3\ 5\ \bar5\ 0\ 2\ \bar1) + q^{-13}(\bar2\ 3\ 1\ \overline{10}\ 9\ 9\ \overline{20}\ 3\ 17\ \overline{11}\ \bar5\ 8\ \bar1\ \bar2\ 1) + q^{2n-15}(1\ 0\ \bar4\ 5\ 3\ \overline{14}\ 11\ 12\ \overline{25}\ 3\ 22\ \overline{14}\ \bar8\ 10\ \bar1\ \bar3\ 1) + q^{4n-11}(1\ \bar3\ 0\ 7\ \bar9\ 0\ 15\ \overline{15}\ \bar6\ 20\ \bar6\ \overline{11}\ 8\ 1\ \bar3\ 1) + q^{6n-5}(\bar1\ 2\ 0\ \bar3\ 4\ \bar1\ \bar5\ 5\ 2\ \bar5\ 1\ 2\ \bar1)$
10_{33}	5	A	$1\bar23\bar4/3/\bar2/\bar1^2\bar234^2$	$q^{-2n}(\bar2\ 1) + (\bar4\ 2) + q^{2n}(\bar2\ 1)$	$q^{-6n-13}(\bar1\ 2\ 1\ \bar4\ 1\ 2\ \bar1) + q^{-4n-11}(\bar3\ 4\ 7\ \overline{11}\ \bar4\ 10\ \bar1\ \bar3\ 1) + q^{-2n-13}(1\ 0\ \bar7\ 7\ 15\ \overline{18}\ \overline{12}\ 17\ 3\ \bar7\ 0\ 1) + q^{-11}(1\ 1\ \overline{10}\ 5\ 24\ \overline{21}\ \overline{21}\ 24\ 5\ \overline{10}\ 1\ 1) + q^{2n-9}(1\ 0\ \bar7\ 3\ 17\ \overline{12}\ \overline{18}\ 15\ 7\ \bar7\ 0\ 1) + q^{4n-5}(1\ \bar3\ \bar1\ 10\ \bar4\ \overline{11}\ 7\ 4\ \bar3) + q^{6n+1}(\bar1\ 2\ 1\ \bar4\ 1\ 2\ \bar1)$
10_{34}	5		$\bar12\overline{34}/1^223\bar3^24/3^3\bar4$	$(\bar1\ 1) + q^{2n}(\bar1\ 1) + q^{4n}(\bar1\ 1)$	$q^{-2n-9}(\bar1\ 1\ 1\ \bar2\ 0\ 1\ \bar1) + q^{-7}(\bar1\ 2\ 2\ \bar4\ 0\ 4\ \bar1\ \bar1\ 1) + q^{2n-3}(2\ 0\ \bar4\ 2\ 4\ \bar2\ \bar1\ 1) + q^{4n-9}(1\ 0\ \bar1\ 2\ 2\ \bar2\ \bar2\ 3\ 3\ \bar2\ \bar1\ 1) + q^{6n-5}(1\ \bar1\ \bar2\ 4\ 0\ \bar6\ 3\ 4\ \bar3\ \bar1\ 1) + q^{8n-1}(1\ \bar2\ 0\ 4\ \bar4\ \bar2\ 5\ \bar1\ \bar2\ 1) + q^{10n+5}(\bar1\ 1\ 0\ \bar2\ 1\ 1\ \bar1)$
10_{35}	6	10_{22}, $\{1\}su(2)_q$	$\overline{12}3\bar5/12^2\overline{34}^25^2/1\overline{23}4/\bar23^2$	$q^{-2n}(\bar1) + (\bar3\ 1) + q^{2n}(\bar3\ 1) + q^{4n}(\bar1)$	$q^{-6n-7}(\bar1) + q^{-4n-9}(1\ 1\ \bar3\ \bar1\ 2) + q^{-2n-9}(\bar1\ 3\ 2\ \bar8\ \bar2\ 5\ 0\ \bar1) + q^{-9}(\bar1\ \bar1\ 6\ 3\ \overline{14}\ \bar3\ 11\ \bar1\ \bar3\ 1) + q^{2n-9}(1\ \bar2\ \bar3\ 11\ 3\ \overline{19}\ 0\ 13\ \bar2\ \bar3\ 1) + q^{4n-3}(\bar3\ 4\ 8\ \overline{14}\ \bar7\ 13\ 0\ \bar4\ 1) + q^{6n-1}(\bar1\ 1\ 5\ \bar4\ \bar7\ 5\ 3\ \bar2) + q^{8n+5}(2\ \bar1\ \bar3\ 1\ 1) + q^{10n+11}(\bar1)$
10_{36}	5		$1^22^23\bar4/3\bar4/3/\bar12\bar3$	$q^{2n}(2\bar1)+q^{4n}(3\bar1)+q^{6n}(2\bar1)$	$q^{2n-7}(\bar1\ 1\ 1\ \bar2\ 1\ 1\ \bar1) + q^{4n-5}(\bar2\ 2\ 5\ \bar5\ \bar2\ 6\ \bar1\ \bar2\ 1) + q^{6n-5}(\bar2\ \bar3\ 7\ 5\ \overline{12}\ \bar1\ 9\ \bar2\ \bar3\ 1) + q^{8n-7}(1\ 0\ \bar4\ \bar1\ 10\ 4\ \overline{14}\ \bar1\ 11\ \bar2\ \bar3\ 1) + q^{10n-3}(1\ \bar2\ \bar5\ 9\ 7\ \overline{15}\ \bar3\ 11\ \bar2\ \bar3\ 1) + q^{12n+1}(1\ \bar3\ 0\ 9\ \bar6\ \bar8\ 8\ 1\ \bar3\ 1) + q^{14n+7}(\bar1\ 2\ 1\ \bar4\ 1\ 2\ \bar1)$
10_{37}	5	A	$1^2\overline{234}^2/12\bar3^2/24$	$q^{-2n}(\bar1\ 1) + (\bar3\ 2) + q^{2n}(\bar1\ 1)$	$q^{-6n-13}(\bar1\ 1\ 1\ \bar2\ 0\ 1\ \bar1) + q^{-4n-11}(\bar2\ 2\ 4\ \bar5\ \bar2\ 5\ \bar1\ \bar2\ 1) + q^{-2n-13}(1\ 0\ \bar6\ 5\ 10\ \overline{12}\ \bar5\ 11\ 0\ \bar4\ 1\ 1) + q^{-11}(1\ 2\ \bar8\ 2\ 19\ \overline{14}\ \overline{14}\ 19\ 2\ \bar8\ 2\ 1) + q^{2n-9}(1\ 1\ \bar4\ 0\ 11\ \bar5\ \overline{12}\ 10\ 5\ \bar6\ 0\ 1) + q^{4n-5}(1\ \bar2\ \bar1\ 5\ \bar2\ \bar5\ 4\ 2\ \bar2) + q^{6n+1}(\bar1\ 1\ 0\ \bar2\ 1\ 1\ \bar1)$
10_{38}	5		$1^2\bar2^23^34^2/\bar34/\bar12\bar3$	$q^{2n}(2\bar1)+q^{4n}(3\bar2)+q^{6n}(2\bar1)$	$q^{2n-7}(\bar1\ 1\ 1\ \bar2\ 1\ 1\ \bar1) + q^{4n-7}(\bar1\ \bar3\ 3\ 5\ \bar7\ \bar2\ 6\ \bar2\ \bar2\ 1) + q^{6n-7}(1\ \bar3\ \bar6\ 11\ 7\ \overline{17}\ 0\ 12\ \bar5\ \bar3\ 2) + q^{8n-7}(1\ 1\ \bar4\ \bar5\ 13\ 8\ \overline{20}\ \bar1\ 16\ \bar5\ \bar5\ 3) + q^{10n-3}(2\ \bar2\ \bar8\ 10\ 11\ \overline{19}\ \bar4\ 15\ \bar3\ \bar5\ 2) + q^{12n+1}(1\ \bar4\ 0\ 11\ \bar7\ \bar9\ 10\ 1\ \bar4\ 1) + q^{14n+7}(\bar1\ 2\ 1\ \bar4\ 1\ 2\ \bar1)$
10_{39}	4		$\bar12^3\bar3/23^3/\bar12$	$q^{2n}(1) + q^{4n}(\bar2\ 2\ \bar1) + q^{6n}(\bar2\ 2\ \bar1)$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-11}(\bar1\ 1\ 1\ \bar3\ 1\ 3\ \bar3\ 0\ 2\ \bar2\ 0\ 1\ \bar1)+q^{8n-9}(\bar3\ 1\ 7\ \bar7\ \bar6\ 1\ 3\ \bar2\ \overline{10}\ 9\ 0\ \bar6\ 5\ 0\ \bar2\ 1)+q^{10n-11}(1\ 0\ \bar5\ 3\ 11\ \overline{10}\ \overline{11}\ 18\ 2\ \overline{17}\ 9\ 4\ \bar9\ 5\ 1\ \bar3\ 1)+q^{12n-7}(1\ \bar3\ \bar1\ 10\ \bar5\ \overline{13}\ 14\ 6\ \overline{16}\ 6\ 6\ \bar9\ 4\ 2\ \bar3\ 1)+q^{14n-1}(\bar1\ 2\ 1\ \bar5\ 2\ 5\ \bar5\ \bar1\ 4\ \bar3\ 0\ 2\ \bar1)$
10_{40}	4	10_{103}, $\{1\}su(n)_q$	$\bar12^2\bar3/23^3/\bar1^22$	$q^{2n}(4\ \bar2\ 1) + q^{4n}(3\ \bar2\ 1)$	$q^{2n-13}(\bar1\ 2\ 1\ \bar6\ 3\ 6\ \bar8\ 1\ 6\ \bar4\ 1\ 2\ \bar1) + q^{4n-11}(\bar2\ 4\ 5\ \overline{15}\ 2\ 24\ \overline{17}\ \overline{11}\ 23\ 5\ \bar9\ 8\ 0\ \bar2\ 1) + q^{6n-13}(1\ 0\ \bar4\ 6\ 8\ \overline{20}\ 0\ 32\ \overline{19}\ \overline{21}\ 28\ \bar2\ \overline{14}\ 9\ 1\ \bar3\ 1) + q^{8n-9}(1\ \bar3\ 0\ 9\ \overline{11}\ \bar9\ 25\ \bar7\ \overline{23}\ 20\ 4\ \overline{14}\ 6\ 2\ \bar3\ 1) + q^{10n-3}(\bar1\ 2\ 0\ \bar5\ 5\ 3\ \bar9\ 3\ 5\ \bar5\ 0\ 2\ \bar1)$
10_{41}	5	10_{94}, $\{1\}su(2)_q$	$1\bar23\bar4/\bar234/\overline{12}34/\bar23^2$	$(1\ \bar1) + q^{2n}(4\ \bar3\ 1) + q^{4n}(1\ \bar1)$	$q^{-2n-11}(\bar1\ 1\ 1\ \bar2\ 0\ 1\ \bar1) + q^{-13}(1\ 1\ \bar5\ 0\ 9\ \bar5\ \bar6\ 7\ \bar1\ \bar3\ 2) + q^{2n-13}(\bar2\ 3\ 5\ \overline{12}\ \bar5\ 20\ \bar4\ \overline{17}\ 11\ 3\ \bar8\ 3\ 1\ \bar1) + q^{4n-13}(1\ \bar3\ 1\ 12\ \overline{13}\ \overline{16}\ 30\ 4\ \overline{30}\ 14\ 10\ \overline{14}\ 5\ 3\ \bar3\ 1) + q^{6n-9}(\bar1\ 0\ 6\ \bar6\ \overline{13}\ 17\ 9\ \overline{23}\ 4\ 12\ \overline{10}\ 0\ 4\ \bar2) + q^{8n-3}(2\ \bar2\ \bar4\ 7\ 3\ \bar8\ 2\ 4\ \bar3\ 0\ 1) + q^{10n+3}(\bar1\ 1\ 1\ \bar2\ 0\ 1\ \bar1)$
10_{42}	5		$1\bar23\bar4/\bar1^22^23\bar4^2/3$	$q^{-2n}(\bar2\ 1) + (\bar5\ 3\ \bar1) + q^{2n}(\bar1\ 1)$	$q^{-6n-13}(\bar1\ 2\ 1\ \bar4\ 1\ 2\ \bar1)+q^{-4n-15}(1\ 0\ \bar6\ 5\ 10\ \overline{14}\ \bar4\ 13\ \bar3\ \bar4\ 2)+q^{-2n-15}(\bar2\ 4\ 3\ \overline{17}\ 7\ 27\ \overline{26}\ \overline{16}\ 28\ \bar2\ \overline{12}\ 5\ 1\ \bar1)+q^{-15}(1\ \bar3\ 2\ 10\ \overline{19}\ \bar3\ 41\ \overline{24}\ \overline{33}\ 38\ 4\ \overline{21}\ 8\ 3\ \bar3\ 1)+q^{2n-11}(\bar1\ 0\ 5\ \bar8\ \bar6\ 22\ \bar7\ \overline{26}\ 19\ 10\ \overline{15}\ 1\ 4\ \bar2)+q^{4n-5}(2\ \bar2\ \bar2\ 8\ \bar2\ \bar8\ 7\ 4\ \bar4\ 0\ 1)+q^{6n+1}(\bar1\ 1\ 0\ \bar2\ 1\ 1\ \bar1)$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_{43}	5	A 10_{91}, $\{1\}su(2)_q$	$\bar{1}2\overline{34}/2\bar{3}^2 4/12\overline{34}/2^2$	$q^{-2n}(\bar{1}\,1)+(\bar{4}\,3\,\bar{1})+q^{2n}(\bar{1}\,1)$	$q^{-6n-13}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})+q^{-4n-15}(1\,0\,\bar{4}\,3\,6\,\bar{7}\,\bar{2}\,7\,\bar{2}\,\bar{2}\,2)+q^{-2n-15}(\bar{2}\,4\,2\,\overline{14}\,7\,18\,\overline{20}\,\bar{7}\,19\,\bar{5}\,\bar{7}\,5\,0\,\bar{1})+q^{-15}(1\,\bar{3}\,2\,9\,\overline{17}\,\bar{1}\,33\,\overline{22}\,\overline{22}\,33\,\bar{1}\,\overline{17}\,9\,2\,\bar{3}\,1)+q^{2n-11}(\bar{1}\,0\,5\,\bar{7}\,\bar{5}\,19\,\bar{7}\,\overline{20}\,18\,7\,\overline{14}\,2\,4\,\bar{2})+q^{4n-5}(2\,\bar{2}\,\bar{2}\,7\,\bar{2}\,\bar{7}\,6\,3\,\bar{4}\,0\,1)+q^{6n+1}(\bar{1}\,1\,0\,\bar{2}\,1\,1\,\bar{1})$
10_{44}	5		$\overline{12}3/\bar{2}34^2/12\bar{3}4/\bar{2}^2 3$	$(1\,\bar{1})+q^{2n}(5\,\bar{3}\,1)+q^{4n}(2\,\bar{1})$	$q^{-2n-11}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})+q^{-13}(1\,1\,\bar{5}\,1\,9\,\bar{6}\,\bar{5}\,8\,\bar{1}\,\bar{3}\,2)+q^{2n-13}(\bar{2}\,3\,5\,\overline{14}\,\bar{3}\,23\,\overline{10}\,\overline{17}\,16\,2\,\overline{10}\,3\,1\,\bar{1})+q^{4n-13}(1\,\bar{3}\,1\,13\,\overline{16}\,\overline{16}\,39\,\bar{2}\,\overline{37}\,24\,12\,\overline{19}\,5\,4\,\bar{3}\,1)+q^{6n-9}(\bar{1}\,0\,7\,\bar{8}\,\overline{14}\,25\,7\,\overline{32}\,10\,17\,\overline{14}\,\bar{1}\,5\,\bar{2})+q^{8n-3}(2\,\bar{3}\,\bar{5}\,11\,2\,\overline{14}\,4\,7\,\bar{4}\,\bar{1}\,1)+q^{10n+3}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})$
10_{45}	5	A	$12\bar{3}/12\overline{34}/\overline{12}3\bar{4}/\bar{2}3^2$	$q^{-2n}(\bar{2}\,1)+(\bar{6}\,3\,\bar{1})+q^{2n}(\bar{2}\,1)$	$q^{-6n-13}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})+q^{-4n-15}(1\,0\,\bar{6}\,6\,10\,\overline{16}\,\bar{4}\,14\,\bar{3}\,\bar{4}\,2)+q^{-2n-15}(\bar{2}\,4\,2\,\overline{18}\,11\,28\,\overline{34}\,\overline{16}\,32\,\bar{3}\,\overline{13}\,5\,1\,\bar{1})+q^{-15}(1\,\bar{3}\,3\,9\,\overline{23}\,3\,48\,\overline{36}\,\overline{36}\,48\,3\,\overline{23}\,9\,3\,\bar{3}\,1)+q^{2n-11}(\bar{1}\,1\,5\,\overline{13}\,\bar{3}\,32\,\overline{16}\,\overline{34}\,28\,11\,\overline{18}\,2\,4\,\bar{2})+q^{4n-5}(2\,\bar{4}\,\bar{3}\,14\,\bar{4}\,\overline{16}\,10\,6\,\bar{6}\,0\,1)+q^{6n+1}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})$
10_{46}	3		$\bar{1}2^5/\bar{1}2^3$	$q^{2n}(1)+q^{4n}(0\,1)+q^{6n}(1\,\bar{2}\,1\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-15}(\bar{1}\,0\,0\,\bar{2}\,\bar{1}\,0\,\bar{3}\,\bar{2}\,\bar{2}\,\bar{3}\,\bar{3}\,\bar{2}\,\bar{4}\,\bar{2}\,\bar{2}\,\bar{2}\,\bar{1}\,0\,\bar{1})+q^{12n-15}(1\,\bar{1}\,\bar{1}\,4\,1\,\bar{3}\,4\,4\,\bar{1}\,3\,4\,1\,4\,3\,0\,3\,2\,\bar{1}\,2\,1\,\bar{1}\,1)+q^{14n-9}(\bar{1}\,1\,1\,\bar{3}\,\bar{1}\,3\,\bar{2}\,\bar{3}\,1\,0\,\bar{2}\,0\,\bar{1}\,\bar{1}\,1\,\bar{1}\,\bar{1}\,1\,\bar{1})$
10_{47}	3		$1^5\bar{2}/1^2\bar{2}^2$	$q^{2n}(1)+q^{4n}(\bar{1}\,3\,\bar{1}\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-17}(\bar{1}\,1\,1\,\bar{3}\,0\,3\,\bar{3}\,\bar{1}\,2\,\bar{1}\,\bar{1}\,1\,\bar{2}\,0\,1\,\bar{2}\,\bar{1}\,1\,\bar{1})+q^{8n-17}(1\,\bar{1}\,0\,5\,\bar{2}\,\bar{4}\,10\,3\,\bar{6}\,7\,7\,\bar{2}\,5\,5\,\bar{1}\,7\,3\,\bar{2}\,4\,1\,\bar{1}\,1)+q^{10n-11}(\bar{1}\,1\,0\,\bar{4}\,1\,3\,\bar{7}\,\bar{3}\,3\,\bar{4}\,\bar{6}\,1\,\bar{4}\,\bar{2}\,1\,\bar{3}\,\bar{1}\,1\,\bar{1})$
10_{48}	3	FA, $\{1\}su(n)_q$	$\bar{1}^2 2^4/\bar{1}^3 2$	$(\bar{2}\,3\,\bar{1}\,1)$	$q^{-2n-21}(\bar{1}\,1\,\bar{1}\,\bar{3}\,3\,0\,\bar{8}\,3\,4\,\bar{9}\,\bar{1}\,4\,\bar{6}\,0\,1\,\bar{3}\,0\,1\,\bar{1})+q^{-21}(1\,\bar{1}\,1\,4\,\bar{3}\,2\,10\,\bar{6}\,0\,19\,\bar{5}\,\bar{6}\,18\,0\,\bar{5}\,9\,0\,\bar{2}\,4\,0\,\bar{1}\,1)+q^{2n-15}(\bar{1}\,1\,\bar{1}\,\bar{3}\,2\,\bar{1}\,\bar{7}\,4\,0\,\bar{9}\,3\,3\,\bar{7}\,1\,2\,\bar{3}\,0\,1\,\bar{1})$
10_{49}	4		$1^4 2^3 3/1\bar{2}3$	$q^{2n}(1)+q^{4n}(0\,1)+q^{6n}(1\,0\,1)+q^{8n}(3\,\bar{2}\,2)+q^{10n}(0\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-7}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{12n-9}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{14n-9}(2\,1\,\bar{5}\,3\,7\,\bar{8}\,\bar{2}\,10\,\bar{4}\,\bar{4}\,7\,\bar{3}\,\bar{2}\,5\,\bar{2}\,\bar{1}\,2)+q^{16n-7}(1\,2\,\bar{7}\,\bar{1}\,14\,\bar{9}\,\overline{14}\,18\,1\,\overline{17}\,11\,2\,\overline{11}\,8\,0\,\bar{5}\,3)+q^{18n-1}(\bar{3}\,\bar{3}\,7\,\bar{3}\,\overline{13}\,6\,6\,\overline{14}\,0\,4\,\bar{9}\,0\,2\,\bar{5})+q^{20n+5}(3\,1\,\bar{2}\,4\,5\,\bar{1}\,2\,4\,0\,2\,2)+q^{22n+11}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})$
10_{50}	4		$\bar{1}2^2\bar{3}/23^2/\bar{1}2^3$	$q^{2n}(1)+q^{4n}(\bar{2}\,2\,\bar{1})+q^{6n}(\bar{2}\,1\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-11}(\bar{1}\,1\,1\,\bar{3}\,1\,3\,\bar{3}\,0\,2\,\bar{2}\,0\,1\,\bar{1})+q^{8n-9}(\bar{3}\,0\,6\,\bar{6}\,\bar{7}\,9\,\bar{1}\,\bar{9}\,6\,0\,\bar{6}\,4\,0\,\bar{2}\,1)+q^{10n-11}(1\,0\,\bar{4}\,2\,9\,\bar{6}\,\overline{10}\,12\,4\,\overline{12}\,4\,4\,\bar{7}\,3\,1\,\bar{2}\,1)+q^{12n-7}(1\,\bar{2}\,\bar{1}\,7\,\bar{1}\,\bar{9}\,8\,7\,\bar{9}\,3\,6\,\bar{5}\,2\,2\,\bar{2}\,1)+q^{14n-1}(\bar{1}\,1\,1\,\bar{3}\,\bar{1}\,3\,\bar{2}\,\bar{2}\,2\,\bar{1}\,\bar{1}\,1\,\bar{1})$
10_{51}	4		$\bar{1}2^2\bar{3}/\bar{1}^2 23^2/2^2$	$q^{2n}(4\,\bar{2}\,1)+q^{4n}(3\,\bar{1}\,1)$	$q^{2n-13}(\bar{1}\,2\,1\,\bar{6}\,3\,6\,\bar{8}\,1\,6\,\bar{4}\,\bar{1}\,2\,\bar{1})+q^{4n-11}(\bar{2}\,3\,5\,\overline{13}\,\bar{1}\,22\,\overline{13}\,\overline{13}\,20\,\bar{3}\,\bar{9}\,7\,0\,\bar{2}\,1)+q^{6n-13}(1\,0\,\bar{3}\,5\,7\,\overline{14}\,\bar{2}\,27\,\overline{10}\,\overline{19}\,21\,3\,\overline{11}\,6\,1\,\bar{2}\,1)+q^{8n-9}(1\,\bar{2}\,0\,7\,\bar{6}\,\bar{8}\,18\,\bar{1}\,\overline{18}\,13\,6\,\bar{9}\,4\,2\,\bar{2}\,1)+q^{10n-3}(\bar{1}\,1\,0\,\bar{4}\,1\,3\,\bar{7}\,\bar{1}\,3\,\bar{3}\,\bar{1}\,1\,\bar{1})$
10_{52}	4		$\bar{2}3/1^3\bar{2}/1^2\bar{2}\bar{3}^2$	$(2\,\bar{1}\,1)+q^{2n}(3\,\bar{2}\,1)$	$q^{-2n-17}(\bar{1}\,1\,1\,\bar{3}\,0\,3\,\bar{3}\,\bar{1}\,2\,\bar{2}\,\bar{1}\,1\,\bar{1})+q^{-17}(1\,\bar{2}\,0\,7\,\bar{4}\,\bar{7}\,12\,2\,\overline{10}\,8\,3\,\bar{6}\,4\,1\,\bar{2}\,1)+q^{2n-15}(1\,\bar{2}\,\bar{2}\,8\,\bar{1}\,\overline{13}\,8\,11\,\overline{13}\,0\,10\,\bar{6}\,0\,5\,\bar{2}\,0\,1)+q^{4n-13}(1\,\bar{2}\,\bar{2}\,8\,\bar{1}\,\overline{13}\,8\,10\,\overline{12}\,\bar{1}\,7\,\bar{5}\,0\,2\,\bar{2})+q^{6n-7}(\bar{1}\,2\,1\,\bar{6}\,2\,7\,\bar{7}\,\bar{2}\,6\,\bar{3}\,\bar{1}\,2\,\bar{1})$
10_{53}	5		$1\bar{2}3^2 4/23^2/\overline{12}34^2/\bar{1}2^2$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{3}3)+q^{8n}(\bar{3}\,2)+q^{10n}(\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(3\,0\,\bar{8}\,8\,8\,\overline{14}\,2\,10\,\bar{6}\,\bar{1}\,3)+q^{12n-5}(2\,4\,\bar{4}\,\overline{12}\,15\,15\,\overline{26}\,\bar{1}\,22\,\bar{9}\,\bar{6}\,6)+q^{14n-3}(1\,5\,\bar{5}\,\overline{17}\,17\,21\,\overline{32}\,\bar{6}\,26\,\bar{8}\,\bar{9}\,5)+q^{16n+1}(1\,\bar{1}\,\overline{13}\,5\,22\,\overline{21}\,\overline{16}\,22\,\bar{2}\,\overline{10}\,3)+q^{18n+7}(\bar{3}\,0\,11\,\bar{4}\,\overline{11}\,8\,4\,\bar{5})+q^{20n+13}(3\,0\,\bar{3}\,2\,2)+q^{22n+19}(\bar{1})$
10_{54}	4		$\bar{1}^2\bar{2}3^2/\bar{2}3^3/\bar{1}2$	$(2\,\bar{1}\,1)+q^{2n}(2\,\bar{1}\,1)$	$q^{-2n-17}(\bar{1}\,1\,1\,\bar{3}\,0\,3\,\bar{3}\,\bar{1}\,2\,\bar{2}\,\bar{1}\,1\,\bar{1})+q^{-17}(1\,\bar{2}\,\bar{1}\,6\,\bar{2}\,\bar{7}\,8\,3\,\bar{8}\,5\,3\,\bar{5}\,3\,1\,\bar{2}\,1)+q^{2n-15}(1\,\bar{1}\,\bar{2}\,5\,3\,\bar{8}\,1\,11\,\bar{4}\,\bar{4}\,8\,\bar{1}\,\bar{1}\,4\,\bar{1}\,0\,1)+q^{4n-13}(1\,\bar{1}\,\bar{1}\,4\,0\,\bar{5}\,2\,4\,\bar{2}\,\bar{2}\,1\,0\,0\,0\,\bar{1})+q^{6n-7}(\bar{1}\,1\,0\,\bar{3}\,1\,2\,\bar{4}\,0\,2\,\bar{2}\,0\,1\,\bar{1})$
10_{55}	5		$1\bar{2}/1^2 23^3 4/2\bar{3}4$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{2}2)+q^{8n}(\bar{3}\,2)+q^{10n}(\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(2\,1\,\bar{5}\,3\,6\,\bar{7}\,0\,6\,\bar{3}\,\bar{1}\,2)+q^{12n-5}(2\,2\,\bar{4}\,\bar{5}\,9\,5\,\overline{13}\,1\,10\,\bar{5}\,\bar{3}\,3)+q^{14n-3}(1\,3\,\bar{6}\,\overline{10}\,16\,8\,\overline{24}\,3\,15\,\bar{8}\,\bar{4}\,3)+q^{16n+1}(1\,\bar{1}\,\overline{10}\,7\,16\,\overline{20}\,\bar{8}\,19\,\bar{4}\,\bar{7}\,3)+q^{18n+7}(\bar{3}\,0\,10\,\bar{5}\,\overline{10}\,8\,3\,\bar{5})+q^{20n+13}(3\,0\,\bar{3}\,2\,2)+q^{22n+19}(\bar{1})$
10_{56}	4	10_{25}, $\{1\}su(n)_q$	$\bar{1}2^2 3^2/\bar{2}3/\bar{1}2^3$	$q^{2n}(1)+q^{4n}(\bar{2}\,2\,\bar{1})+q^{6n}(\bar{3}\,2\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-11}(\bar{1}\,1\,1\,\bar{3}\,1\,3\,\bar{3}\,0\,2\,\bar{2}\,0\,1\,\bar{1})+q^{8n-9}(\bar{3}\,1\,6\,\bar{8}\,\bar{5}\,1\,2\,\bar{4}\,\bar{9}\,9\,\bar{1}\,6\,5\,0\,\bar{2}\,1)+q^{10n-11}(1\,0\,\bar{5}\,4\,12\,\overline{13}\,\overline{10}\,23\,\bar{3}\,\overline{18}\,14\,2\,\overline{11}\,6\,1\,\bar{3}\,1)+q^{12n-7}(1\,\bar{3}\,\bar{1}\,11\,\bar{6}\,\overline{16}\,19\,7\,\overline{22}\,10\,9\,\overline{12}\,4\,3\,\bar{3}\,1)+q^{14n-1}(\bar{1}\,2\,1\,\bar{6}\,2\,7\,7\,\bar{2}\,6\,\bar{3}\,\bar{1}\,2\,\bar{1})$
10_{57}	4		$\bar{1}2^2\bar{3}/\bar{1}^2 23^3/2$	$q^{2n}(4\,\bar{2}\,1)+q^{4n}(4\,\bar{2}\,1)$	$q^{2n-13}(\bar{1}\,2\,1\,\bar{6}\,3\,6\,\bar{8}\,1\,6\,\bar{4}\,\bar{1}\,2\,\bar{1})+q^{4n-11}(\bar{2}\,4\,4\,\overline{15}\,4\,22\,\overline{19}\,\bar{9}\,23\,\bar{6}\,9\,8\,0\,\bar{2}\,1)+q^{6n-13}(1\,0\,\bar{4}\,7\,8\,\overline{23}\,5\,36\,\overline{27}\,\overline{18}\,34\,\bar{5}\,\overline{15}\,10\,1\,\bar{3}\,1)+q^{8n-9}(1\,\bar{3}\,0\,10\,\overline{13}\,\overline{10}\,32\,\overline{11}\,\overline{29}\,27\,5\,\overline{17}\,7\,3\,\bar{3}\,1)+q^{10n-3}(\bar{1}\,2\,0\,\bar{6}\,6\,4\,\overline{13}\,3\,7\,\bar{6}\,\bar{1}\,2\,\bar{1})$
10_{58}	6		$2\overline{345}/1\bar{2}34\bar{5}/4/12\bar{3}$	$q^{-2n}(\bar{1})+(\bar{4}\,1)+q^{2n}(\bar{4}\,2)+q^{4n}(\bar{1})$	$q^{-6n-7}(\bar{1})+q^{-4n-9}(1\,0\,\bar{4}\,\bar{1}\,2)+q^{-2n-7}(6\,1\,\overline{12}\,0\,7\,0\,\bar{1})+q^{-9}(\bar{3}\,\bar{1}\,13\,2\,\overline{24}\,0\,17\,\bar{3}\,\bar{4}\,1)+q^{2n-9}(2\,\bar{5}\,\bar{7}\,22\,4\,\overline{35}\,3\,22\,\bar{7}\,\bar{5}\,2)+q^{4n-7}(1\,0\,\bar{8}\,11\,17\,\overline{27}\,\bar{9}\,25\,\bar{2}\,\bar{7}\,3)+q^{6n-1}(\bar{3}\,1\,10\,\bar{9}\,\overline{13}\,10\,4\,\bar{5})+q^{8n+5}(3\,\bar{1}\,\bar{4}\,2\,2)+q^{10n+11}(\bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_{59}	5	10_{106}, $\{1\}su(2)_q$	$12^2\bar{3}4/\bar{1}^2 2\bar{3}4^2/\overline{23}4$	$(1\bar{1})+q^{2n}(5\bar{3}1)+q^{4n}(1\bar{1})$	$q^{-2n-11}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})+q^{-13}(1\,1\,\bar{5}\,1\,9\,\bar{6}\,\bar{5}\,8\,\bar{1}\,\bar{3}\,2)+q^{2n-13}(\bar{2}\,3\,5\,\overline{14}\,\bar{4}\,23\,\bar{9}\,\overline{18}\,15\,2\,\overline{10}\,3\,1\,\bar{1})+q^{4n-13}(1\,\bar{3}\,1\,13\,\overline{14}\,\overline{18}\,37\,4\,\overline{37}\,20\,14\,\overline{17}\,5\,4\,\bar{3}\,1)+q^{6n-9}(\bar{1}\,0\,6\,\bar{6}\,\overline{15}\,18\,12\,\overline{29}\,2\,16\,\overline{11}\,\bar{2}\,4\,\bar{2})+q^{8n-3}(2\,\bar{2}\,\bar{4}\,7\,4\,\bar{8}\,1\,5\,\bar{2}\,0\,1)+q^{10n+3}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
10_{60}	5	10_{83}, $\{1\}su(2)_q$ 7_6, $\{2\}su(n)_q$	$1\bar{2}3^24/12\bar{3}4/\bar{3}$	$(\bar{5}\,3\,\bar{1})+q^{2n}(\bar{3}\,2)+q^{4n}(\bar{1})$	$q^{2n-7}(\bar{1}\,1\,1\,\bar{2}\,1\,1\,\bar{1})+q^{4n-7}(1\,\bar{2}\,\bar{1}\,6\,\bar{2}\,\bar{4}\,5\,0\,\bar{2}\,1)+q^{6n-3}(\bar{1}\,\bar{1}\,4\,0\,\bar{5}\,2\,2\,\bar{2})+q^{8n+3}(2\,0\,\bar{2}\,1\,1)+q^{10n+9}(\bar{1})$
10_{61}	4		$1\bar{2}3^3/\bar{1}^2\bar{2}3^3$	$q^{2n}(\bar{1}\,0\,\bar{1})+q^{4n}(\bar{1}\,1\,\bar{1})$	$q^{2n-15}(\bar{1}\,0\,0\,\bar{2}\,\bar{2}\,0\,\bar{3}\,\bar{2}\,0\,\bar{2}\,\bar{2}\,0\,\bar{1})+q^{4n-15}(1\,\bar{1}\,\bar{1}\,3\,1\,\bar{3}\,3\,4\,\bar{2}\,2\,3\,\bar{2}\,1\,1\,\bar{1}\,1)+q^{6n-13}(1\,0\,\bar{2}\,1\,3\,\bar{1}\,\bar{3}\,3\,1\,\bar{3}\,1\,1\,\bar{2}\,1\,0\,0\,1)+q^{8n-11}(1\,\bar{1}\,\bar{2}\,2\,1\,\bar{2}\,\bar{1}\,0\,1\,0\,\bar{2}\,0\,1\,\bar{1}\,\bar{1})+q^{10n-5}(\bar{1}\,1\,1\,\bar{2}\,0\,2\,\bar{1}\,\bar{1}\,1\,\bar{1}\,0\,1\,\bar{1})$
10_{62}	3		$\bar{1}^2 2^3/\bar{1}2^4$	$q^{2n}(1)+q^{4n}(\bar{2}\,3\,\bar{1}\,1)$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-17}(\bar{1}\,1\,1\,\bar{3}\,0\,4\,\bar{3}\,\bar{1}\,4\,\bar{2}\,\bar{1}\,3\,\bar{2}\,\bar{1}\,2\,\bar{1}\,\bar{1}\,1\,\bar{1})+q^{8n-17}(1\,\bar{1}\,0\,5\,\bar{2}\,\bar{5}\,10\,3\,\overline{10}\,7\,7\,\bar{7}\,5\,6\,\bar{5}\,5\,4\,\bar{3}\,3\,1\,\bar{1}\,1)+q^{10n-11}(\bar{1}\,1\,0\,\bar{4}\,1\,4\,\bar{7}\,\bar{3}\,6\,\bar{4}\,\bar{6}\,4\,\bar{3}\,\bar{3}\,2\,\bar{2}\,\bar{1}\,1\,\bar{1})$
10_{63}	5		$1^2 23^2 4^2/\bar{1}23^2\bar{4}/2\bar{3}$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{2}2)+q^{8n}(\bar{2}\,2)+q^{10n}(\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-5}(2\,1\,\bar{5}\,3\,6\,\bar{7}\,0\,6\,\bar{3}\,\bar{1}\,2)+q^{12n-5}(2\,2\,\bar{3}\,\bar{5}\,7\,7\,\overline{11}\,\bar{1}\,10\,\bar{4}\,\bar{3}\,3)+q^{14n-3}(1\,3\,\bar{5}\,\bar{9}\,12\,8\,\overline{18}\,0\,12\,\bar{6}\,\bar{3}\,3)+q^{16n+1}(1\,0\,\bar{8}\,5\,14\,\overline{14}\,\bar{7}\,15\,\bar{3}\,\bar{6}\,3)+q^{18n+7}(\bar{3}\,\bar{2}\,7\,\bar{4}\,\bar{9}\,5\,2\,\bar{5})+q^{20n+13}(3\,1\,\bar{2}\,2\,2)+q^{22n+19}(\bar{1})$
10_{64}	3		$\bar{1}^3 2^3/\bar{1}2^3$	$q^{2n}(3\,\bar{3}\,1\,\bar{1})$	$q^{2n-19}(\bar{1}\,1\,0\,\bar{4}\,1\,3\,\bar{8}\,\bar{1}\,5\,\bar{7}\,\bar{2}\,5\,\bar{5}\,\bar{2}\,3\,\bar{2}\,\bar{1}\,1\,\bar{1})+q^{4n-19}(1\,\bar{1}\,0\,4\,\bar{3}\,\bar{2}\,12\,\bar{4}\,\overline{10}\,18\,2\,\overline{15}\,13\,4\,\overline{10}\,7\,2\,\bar{4}\,3\,1\,\bar{1}\,1)+q^{6n-13}(\bar{1}\,1\,0\,\bar{3}\,2\,2\,\bar{8}\,3\,7\,\overline{11}\,\bar{1}\,8\,\bar{6}\,\bar{2}\,3\,\bar{2}\,\bar{1}\,1\,\bar{1})$
10_{65}	4		$12^2 3/\overline{12}3^3/\overline{12}^2 3$	$q^{2n}(3\,\bar{2}\,1)+q^{4n}(3\,\bar{1}\,1)$	$q^{2n-13}(\bar{1}\,2\,1\,\bar{5}\,2\,5\,\bar{5}\,0\,4\,\bar{3}\,0\,2\,\bar{1})+q^{4n-11}(\bar{2}\,3\,4\,\overline{11}\,0\,17\,\bar{9}\,\bar{9}\,14\,\bar{3}\,\bar{6}\,6\,\bar{1}\,\bar{2}\,1)+q^{6n-13}(1\,0\,\bar{3}\,5\,6\,\overline{14}\,\bar{1}\,22\,\overline{12}\,\overline{14}\,17\,\bar{1}\,\bar{9}\,6\,0\,\bar{2}\,1)+q^{8n-9}(1\,\bar{2}\,0\,7\,\bar{6}\,\bar{7}\,18\,\bar{1}\,\overline{15}\,14\,5\,\bar{8}\,5\,2\,\bar{2}\,1)+q^{10n-3}(\bar{1}\,1\,0\,\bar{4}\,1\,3\,\bar{7}\,\bar{1}\,3\,\bar{3}\,\bar{1}\,1\,\bar{1})$
10_{66}	4		$1^2 2\bar{3}/2^2/1\bar{2}3^4/2$	$q^{2n}(1)+q^{4n}(0\ \ 1)+q^{6n}(1\ 0\ 1)+q^{8n}(4\ \bar{3}\ 2)+q^{10n}(1\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\,0\,0\,1)+q^{6n-3}(1\,0\,0\,1\,0\,0\,1)+q^{8n-5}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{10n-7}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{12n-9}(1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1)+q^{14n-9}(2\,1\,\bar{6}\,4\,9\,\overline{13}\,\bar{1}\,16\,\overline{11}\,\bar{6}\,14\,\bar{6}\,\bar{5}\,7\,\bar{2}\,\bar{2}\,2)+q^{16n-7}(1\,2\,\bar{9}\,\bar{1}\,20\,\overline{16}\,\overline{20}\,33\,\bar{5}\,\overline{30}\,25\,3\,\overline{21}\,12\,2\,\bar{7}\,3)+q^{18n-1}(\bar{3}\,\bar{2}\,12\,\bar{4}\,\overline{19}\,18\,10\,\overline{26}\,8\,13\,\overline{16}\,1\,6\,\bar{5})+q^{20n+5}(3\,\bar{1}\,\bar{6}\,6\,5\,\bar{8}\,2\,6\,\bar{3}\,0\,2)+q^{22n+11}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
10_{67}	5		$\bar{1}2\overline{34}/23^24/1^22^2\overline{34}$	$q^{2n}(2\bar{1})+q^{4n}(4\bar{2})+q^{6n}(2\bar{1})$	$q^{2n-7}(\bar{1}\,1\,1\,\bar{2}\,1\,1\,\bar{1})+q^{4n-7}(\bar{1}\,\bar{2}\,4\,4\,\bar{7}\,\bar{1}\,6\,\bar{2}\,\bar{2}\,1)+q^{6n-7}(1\,\bar{4}\,\bar{6}\,15\,6\,\overline{22}\,3\,14\,\bar{6}\,\bar{3}\,2)+q^{8n-7}(1\,1\,\bar{5}\,\bar{7}\,16\,11\,\overline{27}\,\bar{2}\,21\,\bar{6}\,\bar{6}\,3)+q^{10n-3}(2\,\bar{2}\,\bar{9}\,11\,15\,\overline{21}\,\bar{8}\,18\,\bar{2}\,\bar{6}\,2)+q^{12n+1}(1\,\bar{4}\,0\,12\,\bar{7}\,\overline{11}\,10\,2\,\bar{4}\,1)+q^{14n+7}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})$
10_{68}	5		$\bar{1}2\bar{3}4^2/1^2\bar{2}3\bar{4}/2^2\bar{3}^2$	$(\bar{2}\,1)+q^{2n}(\bar{3}\,2)+q^{4n}(\bar{1}\,1)$	$q^{-2n-9}(\bar{1}\,2\,1\,\bar{4}\,1\,2\,\bar{1})+q^{-9}(\bar{1}\,0\,5\,\bar{1}\,\bar{9}\,3\,7\,\bar{3}\,\bar{2}\,1)+q^{2n-9}(1\,\bar{3}\,\bar{1}\,12\,\bar{5}\,\overline{16}\,11\,10\,\bar{8}\,\bar{2}\,2)+q^{4n-9}(1\,1\,\bar{3}\,\bar{2}\,11\,0\,\overline{18}\,7\,14\,\bar{8}\,\bar{4}\,3)+q^{6n-5}(2\,\bar{1}\,\bar{5}\,8\,5\,\overline{15}\,1\,13\,\bar{4}\,\bar{4}\,2)+q^{8n-1}(1\,\bar{3}\,0\,6\,\bar{6}\,\bar{4}\,8\,0\,\bar{3}\,1)+q^{10n+5}(\bar{1}\,1\,0\,\bar{2}\,1\,1\,\bar{1})$
10_{69}	5		$12\bar{3}4/\bar{2}^23/\bar{1}^22^334$	$q^{2n}(5\bar{3}1)+q^{4n}(4\bar{2})+q^{6n}(1)$	$q^{2n-13}(\bar{1}\,3\,0\,\bar{9}\,8\,8\,\overline{15}\,3\,10\,\bar{7}\,\bar{1}\,3\,\bar{1})+q^{4n-13}(1\,\bar{3}\,3\,7\,\overline{21}\,5\,36\,\overline{31}\,\overline{19}\,38\,\bar{7}\,\overline{17}\,11\,1\,\bar{3}\,1)+q^{6n-9}(\bar{2}\,3\,7\,\overline{22}\,1\,45\,\overline{28}\,\overline{36}\,41\,6\,\overline{24}\,6\,5\,\bar{3})+q^{8n-5}(1\,4\,\overline{12}\,\bar{4}\,34\,\overline{12}\,\overline{34}\,26\,12\,\overline{15}\,1\,3)+q^{10n+1}(\bar{3}\,\bar{2}\,12\,\bar{1}\,\bar{1}\,7\,6\,8\,\bar{5}\,\bar{1})+q^{12n+7}(3\,0\,\bar{4}\,1\,2)+q^{14n+13}(\bar{1})$
10_{70}	5		$1\bar{2}3^3\bar{4}/\bar{1}^22^23\bar{4}$	$(0\,\bar{1})+q^{2n}(4\,\bar{3}\,1)+q^{4n}(1\,\bar{1})$	$q^{-2n-11}(\bar{1}\,0\,0\,\bar{1}\,\bar{1}\,0\,\bar{1})+q^{-13}(1\,1\,\bar{3}\,1\,5\,\bar{2}\,\bar{2}\,3\,\bar{1}\,\bar{1}\,2)+q^{2n-13}(\bar{2}\,3\,4\,\overline{12}\,\bar{2}\,16\,\bar{9}\,\overline{10}\,10\,\bar{3}\,\bar{6}\,4\,0\,\bar{1})+q^{4n-13}(1\,\bar{3}\,1\,12\,\overline{13}\,\overline{14}\,31\,0\,\overline{28}\,19\,7\,\overline{15}\,7\,3\,\bar{3}\,1)+q^{6n-9}(\bar{1}\,0\,6\,\bar{6}\,\overline{13}\,17\,8\,\overline{24}\,5\,12\,\overline{11}\,0\,4\,\bar{2})+q^{8n-3}(2\,\bar{2}\,\bar{4}\,7\,3\,\bar{8}\,2\,4\,\bar{3}\,0\,1)+q^{10n+3}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$
10_{71}	5	FA, $\{1\}su(n)_q$ 10_{104}, $\{1\}su(2)_q$ 9_{44}, $\{2\}su(n)_q$	$1^2\overline{23}^24/23\bar{4}^2/\bar{1}23^2$	$q^{-2n}(\bar{1}1)+(\bar{5}3\bar{1})+q^{2n}(\bar{1}1)$	$q^{-2n-3}(\bar{1})+q^{-7}(1\,1\,0\,\bar{1}\,0\,1)+q^{2n-7}(\bar{1}\,\bar{1}\,3\,\bar{2}\,\bar{4}\,2\,1\,\bar{2})+q^{4n-5}(1\,\bar{1}\,1\,3\,\bar{3}\,0\,4\,\bar{1}\,\bar{1}\,1)+q^{6n+1}(\bar{1}\,1\,0\,\bar{2}\,1\,1\,\bar{1})$
10_{72}	4		$1^3\bar{2}/1^22^23^2/\bar{2}^2\bar{3}$	$q^{2n}(1)+q^{4n}(\bar{2}\ \ 2\ \ \bar{1})+q^{6n}(\bar{3}\,3\,\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-11}(\bar{1}\,1\,1\,\bar{3}\,1\,3\,\bar{3}\,0\,2\,\bar{2}\,0\,1\,\bar{1})+q^{8n-9}(\bar{3}\,2\,7\,\bar{9}\,\bar{4}\,15\,\bar{5}\,\bar{9}\,12\,\bar{1}\,\bar{6}\,6\,0\,\bar{2}\,1)+q^{10n-11}(1\,0\,\bar{6}\,5\,14\,\overline{18}\,\overline{10}\,29\,\bar{8}\,\overline{22}\,20\,0\,\overline{14}\,8\,1\,\bar{4}\,1)+q^{12n-7}(1\,\bar{4}\,\bar{1}\,15\,\overline{10}\,\overline{20}\,28\,5\,\overline{30}\,17\,10\,\overline{17}\,7\,4\,\bar{4}\,1)+q^{14n-1}(\bar{1}\,3\,1\,\bar{9}\,5\,9\,\overline{12}\,\bar{1}\,9\,\bar{6}\,\bar{1}\,3\,\bar{1})$
10_{73}	5	10_{86}, $\{1\}su(2)_q$	$1\bar{2}3^34/12\bar{3}4/\bar{2}^2$	$q^{2n}(5\bar{3}1)+q^{4n}(3\bar{2})+q^{6n}(1)$	$q^{2n-13}(\bar{1}\,3\,0\,\bar{9}\,8\,8\,\overline{15}\,3\,10\,\bar{7}\,\bar{1}\,3\,\bar{1})+q^{4n-13}(1\,\bar{3}\,3\,8\,\overline{21}\,2\,37\,\overline{28}\,\overline{21}\,37\,\bar{6}\,\overline{17}\,11\,1\,\bar{3}\,1)+q^{6n-9}(\bar{2}\,2\,7\,\overline{19}\,\bar{4}\,40\,\overline{19}\,\overline{37}\,34\,8\,\overline{23}\,5\,5\,\bar{3})+q^{8n-5}(1\,4\,\bar{9}\,\bar{3}\,27\,\bar{7}\,\overline{26}\,20\,11\,\overline{12}\,1\,3)+q^{10n+1}(\bar{3}\,\bar{2}\,9\,\bar{3}\,\overline{13}\,5\,5\,\bar{5}\,\bar{1})+q^{12n+7}(3\,0\,\bar{3}\,2\,2)+q^{14n+13}(\bar{1})$
10_{74}	5		$\bar{1}2\overline{34}/2\overline{34}/23^24/1^22$	$q^{2n}(3\bar{1})+q^{4n}(4\bar{2})+q^{6n}(1\bar{1})$	$q^{2n-7}(\bar{1}\,2\,1\,\bar{4}\,2\,2\,\bar{1})+q^{4n-7}(\bar{1}\,\bar{3}\,6\,7\,\overline{12}\,\bar{2}\,11\,\bar{2}\,\bar{3}\,1)+q^{6n-7}(1\,\bar{4}\,\bar{9}\,16\,13\,\overline{26}\,\bar{3}\,19\,\bar{5}\,\bar{5}\,2)+q^{8n-7}(1\,1\,\bar{4}\,\overline{10}\,12\,19\,\overline{26}\,\overline{12}\,22\,\bar{3}\,\bar{7}\,3)+q^{10n-3}(2\,\bar{1}\,\bar{8}\,6\,16\,\overline{13}\,\overline{13}\,14\,1\,\bar{5}\,2)+q^{12n+1}(1\,\bar{3}\,\bar{1}\,9\,\bar{2}\,\bar{8}\,6\,2\,\bar{3}\,1)+q^{14n+7}(\bar{1}\,1\,1\,\bar{2}\,0\,1\,\bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_{75}	5		$1\bar{2}34/1\bar{2}^2\bar{3}4/23/\bar{2}$	$(\bar4~3~\bar1)+q^{2n}(\bar3~2)+q^{4n}(\bar1)$	$q^{-2n-15}(\bar1~3~\bar1~\bar7~10~2~\overline{15}~8~8~\bar9~0~3~\bar1)+q^{-15}(1~\bar3~3~3~\overline{15}~15~14~\overline{36}~6~32~\overline{18}~\overline{12}~12~0~\bar3~1)+q^{2n-11}(\bar2~3~3~\overline{15}~15~18~\overline{40}~0~39~\overline{12}~\overline{18}~9~4~\bar3)+q^{4n-7}(1~3~\bar9~5~18~\overline{26}~\overline{10}~30~\bar2~\overline{14}~3~3)+q^{6n-1}(\bar3~0~8~\bar9~\bar9~11~3~\bar6~\bar1)+q^{8n+5}(3~\bar1~\bar3~3~2)+q^{10n+11}(\bar1)$
10_{76}	4		$1\bar{2}3^3/1^2\bar{2}3^3/\overline{12}$	$q^{2n}(1)+q^{4n}(\bar2~1~\bar1)+q^{6n}(\bar3~2~\bar1)$	$q^{2n+1}(1)+q^{4n-1}(1~0~0~1)+q^{6n-11}(\bar1~0~0~\bar2~\bar1~0~\bar3~\bar1~\bar1~\bar2~\bar1~0~\bar1)+q^{8n-9}(\bar2~1~4~\bar4~\bar2~7~\bar2~\bar3~6~\bar1~\bar3~3~0~\bar1~1)+q^{10n-11}(1~0~\bar4~4~9~\overline{12}~\bar4~18~\bar7~\overline{10}~14~\bar2~\bar7~6~0~\bar2~1)+q^{12n-7}(1~\bar3~\bar1~10~\bar7~\overline{13}~19~1~\overline{20}~13~5~\overline{12}~5~2~\bar3~1)+q^{14n-1}(\bar1~2~1~\bar6~3~6~\bar8~0~6~\bar4~\bar1~2~\bar1)$
10_{77}	4		$\bar{1}2^3\bar{3}^2/1^22^2\bar{3}$	$q^{2n}(3~\bar1~1)+q^{4n}(3~\bar2~1)$	$q^{2n-13}(\bar1~1~1~\bar3~0~3~\bar3~0~2~\bar2~\bar1~1~\bar1)+q^{4n-11}(\bar1~3~2~\bar7~3~11~\bar7~\bar2~11~\bar2~\bar2~5~0~\bar1~1)+q^{6n-13}(1~0~\bar3~6~4~\overline{16}~7~19~\overline{20}~\bar6~19~\bar8~\bar6~7~\bar1~\bar2~1)+q^{8n-9}(1~\bar3~0~8~\overline{11}~\bar6~22~\overline{11}~\overline{17}~20~\bar1~\overline{12}~7~1~\bar3~1)+q^{10n-3}(\bar1~2~0~\bar5~5~3~\bar9~3~5~\bar5~0~2~\bar1)$
10_{78}	5		$134/2^334/\bar{1}^2\bar{2}3^2\bar{4}$	$q^{2n}(1)+q^{4n}(\bar3~3~\bar1)+q^{6n}(\bar2~2)+q^{8n}(\bar1)$	$q^{2n+1}(1)+q^{4n-1}(1~0~0~1)+q^{6n-11}(\bar1~2~1~\bar6~4~6~\bar8~1~6~\bar4~0~2~\bar1)+q^{8n-11}(1~\bar3~0~11~\overline{10}~\overline{10}~24~\bar5~\overline{20}~19~3~\overline{13}~7~2~\bar3~1)+q^{10n-7}(\bar2~0~10~\bar7~\overline{17}~21~8~\overline{28}~9~15~\overline{15}~0~6~\bar3)+q^{12n-3}(1~5~\bar5~\bar9~14~6~\overline{16}~4~10~\bar6~\bar1~3)+q^{14n+3}(\bar3~\bar4~6~0~\bar9~1~3~\bar4~\bar1)+q^{16n+9}(3~1~\bar2~2~2)+q^{18n+15}(\bar1)$
10_{79}	3	A	$1^3\bar{2}^2/1^2\bar{2}^3$	$(\bar3~4~\bar1~1)$	$q^{-2n-21}(\bar1~1~\bar1~\bar4~4~0~\overline{13}~7~6~\overline{18}~18~\overline{12}~\bar1~3~\bar4~\bar1~1~\bar1)+q^{-21}(1~\bar1~1~5~\bar4~2~16~\overline{11}~\bar1~34~\overline{12}~\overline{12}~34~\bar1~\overline{11}~16~2~\bar4~5~1~\bar1~1)+q^{2n-15}(\bar1~1~\bar1~\bar4~3~\bar1~\overline{12}~8~1~\overline{18}~6~7~\overline{13}~0~4~\bar4~\bar1~1~\bar1)$
10_{80}	4		$1^22\bar{3}/2^2/\bar{1}2^33/2^2$	$q^{2n}(1)+q^{4n}(0~1)+q^{6n}(1~0~1)+q^{8n}(4~\bar3~2)+q^{10n}(0~\bar1)$	$q^{2n+1}(1)+q^{4n-1}(1~0~0~1)+q^{6n-3}(1~0~0~1~0~0~1)+q^{8n-5}(1~0~0~1~0~0~1~0~0~1)+q^{10n-7}(1~0~0~1~0~0~1~0~0~1~0~0~1)+q^{12n-9}(1~0~0~1~0~0~1~0~0~1~0~0~1~0~0~1)+q^{14n-9}(2~1~\bar6~4~9~\overline{13}~\bar1~16~\overline{11}~\bar6~14~\bar6~\bar5~7~\bar2~\bar2~2)+q^{16n-7}(1~2~\bar9~\bar2~20~\overline{14}~\overline{23}~31~\bar1~\overline{32}~22~5~\overline{21}~11~2~\bar7~3)+q^{18n-1}(\bar3~\bar3~10~\bar1~\overline{19}~11~15~\overline{23}~1~14~\overline{13}~\bar1~5~\bar5)+q^{20n+5}(3~1~\bar3~3~6~\bar2~0~5~0~1~2)+q^{22n+11}(\bar1~0~0~\bar1~\bar1~0~\bar1)$
10_{81}	5	A 10_{109}, $\{1\}su(2)_q$	$\bar{1}^2\bar{2}^2\bar{3}/\bar{1}24^2/3^24$	$q^{-2n}(\bar1~1)+(\bar5~4~\bar1)+q^{2n}(\bar1~1)$	$q^{-6n-13}(\bar1~1~1~\bar2~0~1~\bar1)+q^{-4n-15}(1~\bar1~\bar5~5~6~\overline{10}~\bar1~8~\bar3~\bar2~2)+q^{-2n-15}(\bar2~6~2~\overline{20}~14~25~\overline{33}~\bar6~29~\overline{10}~\bar9~7~0~\bar1)+q^{-15}(1~\bar4~3~13~\overline{26}~\bar2~51~\overline{34}~\overline{34}~51~\bar2~\overline{26}~13~3~\bar4~1)+q^{2n-11}(\bar1~0~7~\bar9~\overline{10}~29~\bar6~\overline{33}~25~14~\overline{20}~2~6~\bar2)+q^{4n-5}(2~\bar2~\bar3~8~\bar1~\overline{10}~6~5~\bar5~\bar1~1)+q^{6n+1}(\bar1~1~0~\bar2~1~1~\bar1)$
10_{82}	3		$1^5\bar{2}^3/\bar{1}^22^2$	$q^{2n}(4~\bar3~2~\bar1)$	$q^{2n-19}(\bar1~2~0~\bar6~5~6~\overline{11}~0~11~\bar7~\bar3~8~\bar5~\bar1~5~\bar4~0~2~\bar1)+q^{4n-19}(1~\bar2~0~6~\bar8~\bar4~18~\bar9~\overline{18}~23~3~\overline{23}~15~5~\overline{14}~10~0~\bar7~5~0~\bar2~1)+q^{6n-13}(\bar1~2~0~\bar5~6~4~\overline{12}~4~12~\overline{13}~\bar2~11~\bar7~\bar1~5~\bar4~0~2~\bar1)$
10_{83}	4	10_{60}, $\{1\}su(2)_q$	$\bar{1}2\bar{3}/2\bar{3}^2/2/1^22\bar{3}$	$(\bar4~3~\bar1)+q^{2n}(\bar3~2~\bar1)$	$q^{-2n-15}(\bar1~3~\bar1~\bar8~11~4~\overline{17}~7~9~\bar9~0~3~\bar1)+q^{-13}(\bar2~4~1~\overline{16}~16~17~\overline{36}~3~31~\overline{17}~\overline{10}~12~\bar1~\bar3~1)+q^{2n-15}(1~\bar1~\bar4~8~2~\overline{23}~17~24~\overline{42}~\bar2~37~\overline{16}~\overline{14}~12~0~\bar3~1)+q^{4n-11}(1~\bar3~1~9~\overline{14}~\bar2~28~\overline{21}~\overline{17}~31~\bar4~\overline{16}~9~2~\bar3~1)+q^{6n-5}(\bar1~2~0~\bar5~6~2~\overline{10}~5~5~\bar6~0~2~\bar1)$
10_{84}	4		$1^32^2\bar{3}/2\bar{3}/\bar{1}2\bar{3}$	$q^{2n}(4~\bar2~1)+q^{4n}(4~\bar3~1)$	$q^{2n-13}(\bar1~2~1~\bar6~3~6~\bar8~1~6~\bar4~\bar1~2~\bar1)+q^{4n-11}(\bar1~5~2~\overline{15}~8~21~\overline{21}~\bar6~23~\bar7~\bar8~8~0~\bar2~1)+q^{6n-13}(1~\bar1~\bar5~10~6~\overline{32}~11~40~\overline{40}~\overline{17}~40~\overline{12}~\overline{15}~12~0~\bar3~1)+q^{8n-9}(1~\bar4~1~14~\overline{19}~\overline{12}~44~\overline{16}~\overline{37}~37~4~\overline{23}~10~3~\bar4~1)+q^{10n-3}(\bar1~3~0~\bar9~9~7~\overline{17}~4~11~\bar8~\bar1~3~\bar1)$
10_{85}	3		$1^3\bar{2}^2/\bar{1}^22^5$	$q^{2n}(1)+q^{4n}(\bar3~3~\bar2~1)$	$q^{2n+1}(1)+q^{4n-1}(1~0~0~1)+q^{6n-17}(\bar1~2~2~\bar5~1~7~\bar5~\bar3~7~\bar2~\bar3~4~\bar2~\bar2~4~\bar2~\bar1~2~\bar1)+q^{8n-17}(1~\bar2~\bar1~7~\bar5~\overline{10}~14~3~\overline{18}~8~10~\overline{13}~3~7~\overline{10}~5~4~\bar7~4~1~\bar2~1)+q^{10n-11}(\bar1~2~1~\bar6~3~8~\bar9~\bar4~11~\bar3~\bar7~7~\bar2~\bar3~5~\bar3~\bar1~2~\bar1)$
10_{86}	4		$1^22\bar{3}/2^2\bar{3}/2\bar{3}/\bar{1}2$	$q^{2n}(4~\bar3~1)+q^{4n}(3~\bar2~1)$	$q^{2n-13}(\bar1~3~1~\bar9~5~9~\overline{12}~0~9~\bar6~\bar1~3~\bar1)+q^{4n-11}(\bar2~5~6~\overline{19}~1~31~\overline{20}~\overline{18}~28~\bar5~\overline{13}~10~0~\bar3~1)+q^{6n-13}(1~\bar1~\bar5~8~9~\overline{25}~\bar3~40~\overline{19}~\overline{29}~31~1~\overline{17}~9~1~\bar3~1)+q^{8n-9}(1~\bar3~0~11~\overline{11}~\overline{14}~29~\bar1~\overline{30}~20~9~\overline{16}~6~3~\bar3~1)+q^{10n-3}(\bar1~2~1~\bar6~3~7~\bar9~\bar1~7~\bar4~\bar1~2~\bar1)$
10_{87}	4		$1^32\bar{3}/\bar{1}2\bar{3}/2\bar{3}^2$	$(\bar3~2~\bar1)+q^{2n}(\bar3~3~\bar1)$	$q^{-2n-15}(\bar1~2~0~\bar5~5~3~\bar9~3~5~\bar5~0~2~\bar1)+q^{-13}(\bar1~3~\bar2~\bar7~13~3~\overline{20}~8~15~\overline{12}~\bar4~8~\bar1~\bar2~1)+q^{2n-15}(1~\bar1~\bar4~9~\bar2~\overline{20}~26~8~\overline{41}~16~26~\overline{24}~\bar6~12~\bar2~\bar3~1)+q^{4n-11}(1~\bar4~2~10~\overline{18}~4~28~\overline{30}~\bar8~36~\overline{12}~\overline{16}~13~1~\bar4~1)+q^{6n-5}(\bar1~3~\bar1~\bar6~9~\bar1~\overline{12}~9~5~\bar9~1~3~\bar1)$
10_{88}	5	A	$1^2\bar{2}3/1\bar{2}^2\bar{3}4/234/\bar{1}2\bar{3}\bar{4}$	$q^{-2n}(\bar2~1)+(\bar7~4~\bar1)+q^{2n}(\bar2~1)$	$q^{-6n-13}(\bar1~2~1~\bar4~1~2~\bar1)+q^{-4n-15}(1~\bar1~\bar6~9~9~\overline{19}~\bar2~15~\bar4~\bar4~2)+q^{-2n-15}(\bar2~6~1~\overline{25}~21~35~\overline{51}~\overline{13}~44~\bar9~\overline{15}~7~1~\bar1)+q^{-15}(1~\bar4~4~13~\overline{34}~2~70~\overline{52}~\overline{52}~70~2~\overline{34}~13~4~\bar4~1)+q^{2n-11}(\bar1~1~7~\overline{15}~\bar9~44~\overline{13}~\overline{51}~35~21~\overline{25}~1~6~\bar2)+q^{4n-5}(2~\bar4~\bar4~15~\bar2~\overline{19}~9~9~\bar6~\bar1~1)+q^{6n+1}(\bar1~2~1~\bar4~1~2~\bar1)$
10_{89}	5		$12\bar{3}4/2/\bar{1}2\bar{3}4/2\bar{3}4$	$q^{2n}(6~\bar4~1)+q^{4n}(4~\bar2)+q^{6n}(1)$	$q^{2n-13}(\bar1~4~\bar1~\overline{13}~14~11~\overline{25}~5~16~\overline{11}~\bar2~4~\bar1)+q^{4n-13}(1~\bar4~4~11~\overline{31}~4~56~\overline{45}~\overline{33}~58~\bar8~\overline{27}~15~2~\bar4~1)+q^{6n-9}(\bar2~3~10~\overline{25}~\bar8~61~\overline{23}~\overline{59}~50~19~\overline{34}~5~8~\bar3)+q^{8n-5}(1~4~\overline{12}~\bar7~36~\bar6~\overline{42}~22~19~\overline{16}~\bar2~3)+q^{10n+1}(\bar3~\bar2~12~0~\overline{17}~5~9~\bar4~\bar1)+q^{12n+7}(3~0~\bar4~1~2)+q^{14n+13}(\bar1)$
10_{90}	4		$\bar{1}23^2/1^2\bar{2}^2/1^2\bar{2}^2\bar{3}$	$(\bar4~2~\bar1)+q^{2n}(\bar3~2~\bar1)$	$q^{-2n-15}(\bar1~2~\bar1~\bar6~7~3~\overline{13}~4~6~\bar6~0~2~\bar1)+q^{-13}(\bar1~3~\bar1~\overline{10}~13~10~\overline{26}~3~21~\overline{12}~\bar6~8~\bar1~\bar2~1)+q^{2n-15}(1~\bar1~\bar3~8~0~\overline{19}~19~16~\overline{36}~4~28~\overline{16}~\bar8~10~\bar1~\bar2~1)+q^{4n-11}(1~\bar3~1~8~\overline{14}~\bar1~25~\overline{21}~\overline{14}~28~\bar6~\overline{14}~9~1~\bar3~1)+q^{6n-5}(\bar1~2~0~\bar5~6~2~\overline{10}~5~5~\bar6~0~2~\bar1)$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_{91}	3	FA, $\{1\}su(n)_q$ 10_{43}, $\{1\}su(2)_q$	$1^4\bar{2}^3/\bar{1}^3 2^2$	$(\bar{4}\ 4\ \bar{2}\ 1)$	$q^{-2n-21}(\bar{1}\ 2\ \bar{1}\ \bar{5}\ 8\ 1\ \overline{16}\ 11\ 11\ \overline{21}\ 4\ 13\ \overline{13}\ 1\ 6\ \bar{5}\ 0\ 2\ \bar{1}) + q^{-21}(1\ \bar{2}\ 1\ 6\ \bar{9}\ 1\ 19\ \overline{20}\ \bar{7}\ 39\ \overline{21}\ \overline{22}\ 38\ \bar{7}\ \overline{19}\ 18\ \bar{1}\ \bar{8}\ 6\ 0\ \bar{2}\ 1) + q^{2n-15}(\bar{1}\ 2\ \bar{1}\ \bar{5}\ 7\ 0\ \overline{14}\ 13\ 5\ \overline{21}\ 10\ 11\ \overline{15}\ 2\ 7\ \bar{5}\ 0\ 2\ \bar{1})$
10_{92}	4		$1^3 2^3 3/1\bar{2}3^2/\overline{12}^2\bar{3}$	$q^{2n}(1) + q^{4n}(\bar{3}\ 3\ \bar{1}) + q^{6n}(\bar{4}\ 3\ \bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1\ 0\ 0\ 1) + q^{6n-11}(\bar{1}\ 2\ 1\ \bar{6}\ 4\ 6\ \bar{8}\ 1\ 6\ \bar{4}\ 0\ 2\ \bar{1}) + q^{8n-9}(\bar{4}\ 3\ 11\ \overline{17}\ \bar{7}\ 30\ \overline{13}\ \overline{21}\ 25\ \bar{1}\ \overline{14}\ 9\ 1\ \bar{3}\ 1) + q^{10n-11}(1\ 0\ \bar{7}\ 5\ 20\ \overline{22}\ \overline{22}\ 45\ \bar{2}\ \overline{43}\ 28\ 12\ \overline{25}\ 8\ 5\ \bar{5}\ 1) + q^{12n-7}(1\ \bar{4}\ \bar{1}\ 17\ \overline{10}\ \overline{28}\ 32\ 15\ \overline{41}\ 13\ 20\ \overline{20}\ 3\ 6\ \bar{4}\ 1) + q^{14n-1}(\bar{1}\ 3\ 1\ \overline{10}\ 5\ 12\ \overline{13}\ \bar{3}\ 11\ \bar{5}\ \bar{2}\ 3\ \bar{1})$
10_{93}	4		$\bar{1}^2\bar{2}3/1\bar{2}3^2/\bar{2}3^2$	$(2\ \bar{2}\ 1) + q^{2n}(3\ \bar{2}\ 1)$	$q^{-2n-17}(\bar{1}\ 2\ 2\ \bar{5}\ 1\ 6\ \bar{5}\ \bar{2}\ 5\ \bar{3}\ \bar{1}\ 2\ \bar{1}) + q^{-17}(1\ \bar{3}\ \bar{1}\ 10\ \bar{6}\ \overline{13}\ 17\ 4\ \overline{19}\ 10\ 6\ \overline{11}\ 5\ 2\ \bar{3}\ 1) + q^{2n-15}(1\ \bar{2}\ \bar{3}\ 9\ 2\ \overline{18}\ 6\ 19\ \overline{16}\ 7\ 16\ \bar{7}\ \bar{4}\ 7\ \bar{2}\ \bar{1}\ 1) + q^{4n-13}(1\ \bar{2}\ \bar{2}\ 8\ 0\ \overline{13}\ 6\ 12\ \bar{9}\ \bar{4}\ 7\ \bar{2}\ \bar{1}\ 2\ \bar{1}) + q^{6n-7}(\bar{1}\ 2\ 1\ \bar{6}\ 2\ 7\ \bar{7}\ \bar{2}\ 6\ \bar{3}\ \bar{1}\ 2\ \bar{1})$
10_{94}	3	10_{41}, $\{1\}su(2)_q$	$\bar{1}^3 2^3/1^2\bar{2}^4$	$q^{2n}(4\ \bar{4}\ 2\ \bar{1})$	$q^{-6n-23}(\bar{1}\ 2\ \bar{1}\ \bar{4}\ 7\ \bar{3}\ \overline{10}\ 16\ \bar{3}\ \overline{18}\ 16\ 5\ \overline{16}\ 5\ 6\ \bar{6}\ 0\ 2\ \bar{1}) + q^{-4n-23}(1\ \bar{2}\ 1\ 5\ \bar{9}\ 3\ 13\ \overline{21}\ 8\ 22\ \overline{32}\ 4\ 32\ \overline{23}\ \overline{10}\ 23\ \bar{5}\ \bar{8}\ 7\ 0\ \bar{2}\ 1) + q^{-2n-17}(\bar{1}\ 2\ \bar{1}\ \bar{4}\ 7\ \bar{3}\ \bar{8}\ 12\ \bar{5}\ \overline{12}\ 14\ 0\ \overline{15}\ 7\ 5\ \bar{7}\ 0\ 2\ \bar{1})$
10_{95}	4		$\bar{1}^2 2\bar{3}/2/\bar{1}2^2 3^2/2$	$q^{2n}(5\ \bar{3}\ 1) + q^{4n}(4\ \bar{2}\ 1)$	$q^{2n-11}(1\ \bar{3}\ 1\ 6\ \bar{8}\ 0\ 9\ \bar{6}\ \bar{1}\ 3\ \bar{1}) + q^{4n-11}(\bar{1}\ 3\ \bar{1}\ \bar{7}\ 10\ 1\ \overline{12}\ 12\ 1\ \overline{10}\ 7\ 1\ \bar{3}\ 1) + q^{6n-7}(1\ 1\ \bar{4}\ \bar{2}\ 11\ \bar{1}\ \overline{10}\ 5\ 3\ \bar{2}) + q^{8n-7}(\bar{1}\ 2\ 0\ \bar{5}\ 3\ 5\ \bar{3}\ \bar{2}\ 0\ 1\ 3\ \bar{1}\ \bar{1}\ 1) + q^{10n-1}(1\ \bar{2}\ 0\ 3\ \bar{3}\ \bar{2}\ 2\ \bar{2}\ 0\ 1\ \bar{1})$
10_{96}	5		$12\bar{3}4/23/1\overline{23}^2 4/\bar{3}$	$(\bar{6}\ 3\ \bar{1}) + q^{2n}(\bar{4}\ 2) + q^{4n}(\bar{1})$	$q^{-2n-15}(\bar{1}\ 3\ \bar{3}\ \bar{7}\ 15\ \bar{1}\ \overline{23}\ 12\ 10\ \overline{11}\ 0\ 3\ \bar{1}) + q^{-15}(1\ \bar{3}\ 5\ 2\ \overline{19}\ 26\ 19\ \overline{56}\ 8\ 46\ \overline{23}\ \overline{14}\ 14\ 0\ \bar{3}\ 1) + q^{2n-11}(\bar{2}\ 4\ 2\ \overline{23}\ 21\ 30\ \overline{59}\ \overline{10}\ 55\ \overline{13}\ \overline{24}\ 10\ 4\ \bar{3}) + q^{4n-7}(1\ 3\ \overline{12}\ 6\ 30\ \overline{31}\ \overline{22}\ 39\ 3\ \overline{17}\ 3\ 3) + q^{6n-1}(\bar{3}\ 0\ 11\ \bar{8}\ \overline{15}\ 11\ 6\ \bar{6}\ \bar{1}) + q^{8n+5}(3\ \bar{1}\ \bar{4}\ 2\ 2) + q^{10n+11}(\bar{1})$
10_{97}	5		$\bar{1}2^2 3\bar{4}/12^2\bar{3}/\bar{1}23^2\bar{4}$	$q^{2n}(3\bar{1})+q^{4n}(6\bar{3})+q^{6n}(3\bar{1})$	$q^{2n-7}(\bar{1}\ 2\ 1\ \bar{4}\ 2\ 2\ \bar{1}) + q^{4n-7}(\bar{2}\ \bar{2}\ 9\ 4\ \overline{15}\ 1\ 12\ \bar{3}\ \bar{3}\ 1) + q^{6n-7}(2\ \bar{8}\ \overline{10}\ 31\ 7\ \overline{45}\ 9\ 28\ \overline{12}\ \bar{6}\ 3) + q^{8n-7}(1\ 3\ \bar{8}\ \overline{16}\ 34\ 24\ \overline{58}\ \bar{3}\ 45\ \overline{12}\ \overline{12}\ 6) + q^{10n-3}(3\ \bar{3}\ \overline{17}\ 19\ 32\ \overline{42}\ \overline{21}\ 37\ \bar{1}\ \overline{13}\ 3) + q^{12n+1}(1\ \bar{6}\ 1\ 21\ \overline{12}\ \overline{21}\ 17\ 6\ \bar{6}\ 1) + q^{14n+7}(\bar{1}\ 3\ 1\ \bar{7}\ 1\ 3\ \bar{1})$
10_{98}	4		$\bar{1}2^2 3^2/2/\bar{1}2^2\bar{3}/2$	$q^{2n}(1) + q^{4n}(\bar{3}\ 3\ \bar{1}) + q^{6n}(\bar{4}\ 2\ \bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1\ 0\ 0\ 1) + q^{6n-11}(\bar{1}\ 2\ 1\ \bar{6}\ 4\ 6\ \bar{8}\ 1\ 6\ \bar{4}\ 0\ 2\ \bar{1}) + q^{8n-9}(\bar{4}\ 2\ 11\ \overline{15}\ \bar{9}\ 27\ \overline{10}\ \overline{21}\ 22\ 0\ \overline{14}\ 8\ 1\ \bar{3}\ 1) + q^{10n-11}(1\ 0\ \bar{6}\ 3\ 17\ \overline{16}\ \overline{23}\ 34\ 3\ \overline{38}\ 18\ 13\ \overline{22}\ 5\ 5\ \bar{4}\ 1) + q^{12n-7}(1\ \bar{3}\ \bar{1}\ 13\ \bar{5}\ \overline{21}\ 23\ 16\ \overline{28}\ 8\ 18\ \overline{13}\ 1\ 5\ \bar{3}\ 1) + q^{14n-1}(\bar{1}\ 2\ 1\ \bar{7}\ 2\ 8\ \bar{9}\ \bar{3}\ 7\ \bar{3}\ \bar{2}\ 2\ \bar{1})$
10_{99}	3	A	$1^3 2^3/\bar{1}^3\bar{2}^3$	$(\bar{4}\ 5\ \bar{2}\ 1)$	$q^{-2n-21}(\bar{1}\ 2\ \bar{1}\ \bar{6}\ 9\ 1\ \overline{21}\ 14\ 12\ \overline{29}\ 5\ 16\ \overline{19}\ 0\ 8\ \bar{6}\ \bar{1}\ 2\ \bar{1}) + q^{-21}(1\ \bar{2}\ 1\ 7\ \overline{10}\ 1\ 25\ \overline{24}\ \bar{7}\ 53\ \overline{25}\ \overline{25}\ 53\ \bar{7}\ \overline{24}\ 25\ 1\ \overline{10}\ 7\ 1\ \bar{2}\ 1) + q^{2n-15}(\bar{1}\ 2\ \bar{1}\ \bar{6}\ 8\ 0\ \overline{19}\ 16\ 5\ \overline{29}\ 12\ 14\ \overline{21}\ 1\ 9\ \bar{6}\ \bar{1}\ 2\ \bar{1})$
10_{100}	3	$\{2\}su(2)_q$ DADW	$1^3\bar{2}/1^2\bar{2}1^2\bar{2}$	$q^{2n}(1) + q^{4n}(\bar{3}\ 4\ \bar{2}\ 1)$	$q^{2n+1}(1) + q^{4n-1}(1\ 0\ 0\ 1) + q^{6n-17}(\bar{1}\ 2\ 2\ \bar{6}\ 1\ 9\ \bar{7}\ \bar{4}\ 11\ \bar{3}\ \bar{6}\ 8\ \bar{2}\ \bar{4}\ 5\ \bar{2}\ \bar{2}\ 2\ \bar{1}) + q^{8n-17}(1\ \bar{2}\ \bar{1}\ 8\ \bar{5}\ \overline{12}\ 18\ 5\ \overline{24}\ 12\ 16\ \overline{20}\ 5\ 14\ \overline{14}\ 6\ 8\ \bar{8}\ 4\ 2\ \bar{2}\ 1) + q^{10n-11}(\bar{1}\ 2\ 1\ \bar{7}\ 3\ 10\ \overline{12}\ \bar{6}\ 15\ \bar{5}\ \overline{12}\ 10\ \bar{3}\ \bar{6}\ 6\ \bar{3}\ \bar{2}\ 2\ \bar{1})$
10_{101}	5		$1^2\bar{2}3/2^3 3^2 4/\bar{1}2\bar{3}4$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{3}3)+q^{8n}(\bar{4}\ 3) + q^{10n}(\bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1\ 0\ 0\ 1) + q^{6n-3}(1\ 0\ 0\ 1\ 0\ 0\ 1) + q^{8n-5}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1) + q^{10n-5}(3\ 0\ \bar{8}\ 8\ 8\ \overline{14}\ 2\ 10\ \bar{6}\ \bar{1}\ 3) + q^{12n-5}(3\ 4\ \bar{8}\ \overline{10}\ 21\ 9\ \overline{29}\ 5\ 21\ \overline{11}\ \bar{5}\ 6) + q^{14n-3}(2\ 8\ \overline{10}\ \overline{22}\ 33\ 20\ \overline{49}\ 5\ 33\ \overline{16}\ \bar{8}\ 7) + q^{16n+1}(3\ \bar{1}\ \overline{22}\ 12\ 35\ \overline{38}\ \overline{19}\ 37\ \bar{7}\ \overline{14}\ 6) + q^{18n+7}(\bar{6}\ \bar{2}\ 18\ \bar{7}\ \overline{20}\ 13\ 6\ \bar{9}) + q^{20n+13}(4\ 1\ \bar{4}\ 2\ 3) + q^{22n+19}(\bar{1})$
10_{102}	4		$1^2 2^2\bar{3}/2\bar{3}/\bar{1}2\bar{3}^2$	$(\bar{3}\ 2\ \bar{1}) + q^{2n}(\bar{3}\ 2\ \bar{1})$	$q^{-2n-15}(\bar{1}\ 2\ 0\ \bar{5}\ 5\ 3\ \bar{9}\ 3\ 5\ \bar{5}\ 0\ 2\ \bar{1}) + q^{-13}(\bar{1}\ 2\ \bar{1}\ \bar{7}\ 10\ 5\ \overline{19}\ 4\ 15\ \overline{10}\ \bar{5}\ 7\ \bar{1}\ \bar{2}\ 1) + q^{2n-15}(1\ \bar{1}\ \bar{3}\ 7\ \bar{1}\ \overline{16}\ 19\ 10\ \overline{33}\ 9\ 23\ \overline{16}\ \bar{6}\ 9\ \bar{1}\ \bar{2}\ 1) + q^{4n-11}(1\ \bar{3}\ 1\ 8\ \overline{13}\ 0\ 23\ \overline{21}\ \overline{11}\ 27\ \bar{7}\ \overline{13}\ 9\ 1\ \bar{3}\ 1) + q^{6n-5}(\bar{1}\ 2\ 0\ \bar{5}\ 6\ 2\ \overline{10}\ 5\ 5\ \bar{6}\ 0\ 2\ \bar{1})$
10_{103}	4	10_{40}, $\{1\}su(n)_q$	$1^2 2^2\bar{3}/2^2\bar{3}/\bar{1}2\bar{3}$	$q^{2n}(4\ \bar{2}\ 1) + q^{4n}(3\ \bar{2}\ 1)$	$q^{2n-13}(\bar{1}\ 2\ 1\ \bar{6}\ 3\ 6\ \bar{8}\ 1\ 6\ \bar{4}\ \bar{1}\ 2\ \bar{1}) + q^{4n-11}(\bar{1}\ 4\ 3\ \overline{13}\ 3\ 21\ \overline{15}\ \overline{10}\ 20\ \bar{4}\ \bar{8}\ 7\ 0\ \bar{2}\ 1) + q^{6n-13}(1\ \bar{1}\ \bar{4}\ 8\ 6\ \overline{22}\ 3\ 32\ \overline{21}\ \overline{19}\ 27\ \bar{3}\ \overline{11}\ 8\ 0\ \bar{2}\ 1) + q^{8n-9}(1\ \bar{3}\ 0\ 10\ \overline{11}\ \overline{12}\ 27\ \bar{4}\ \overline{27}\ 20\ 6\ \overline{15}\ 6\ 2\ \bar{3}\ 1) + q^{10n-3}(\bar{1}\ 2\ 1\ \bar{6}\ 3\ 7\ \bar{9}\ \bar{1}\ 7\ \bar{4}\ \bar{1}\ 2\ \bar{1})$
10_{104}	3	FA, $\{1\}su(n)_q$ 10_{71}, $\{1\}su(2)_q$	$\bar{1}^2 2^3/\bar{1}^2 2\bar{1}2$	$(\bar{5}\ 4\ \bar{2}\ 1)$	$q^{-2n-21}(\bar{1}\ 2\ \bar{1}\ \bar{5}\ 9\ 1\ \overline{18}\ 14\ 13\ \overline{26}\ 6\ 16\ \overline{15}\ 1\ 7\ \bar{5}\ 0\ 2\ \bar{1}) + q^{-21}(1\ \bar{2}\ 1\ 6\ \overline{10}\ 1\ 21\ \overline{24}\ \bar{9}\ 46\ \overline{27}\ \overline{28}\ 45\ \bar{9}\ \overline{23}\ 20\ \bar{1}\ \bar{9}\ 6\ 0\ \bar{2}\ 1) + q^{2n-15}(\bar{1}\ 2\ \bar{1}\ \bar{5}\ 8\ 0\ \overline{16}\ 16\ 7\ \overline{26}\ 12\ 14\ \overline{17}\ 2\ 8\ \bar{5}\ 0\ 2\ \bar{1})$
10_{105}	5		$\bar{1}2\bar{3}4/\overline{12}3/\bar{2}3^2 4^2$	$(1\ \bar{1}) + q^{2n}(6\ \bar{4}\ 1) + q^{4n}(2\ \bar{1})$	$q^{-2n-11}(\bar{1}\ 1\ 1\ \bar{2}\ 0\ 1\ \bar{1}) + q^{-13}(1\ 0\ \bar{6}\ 3\ 9\ \bar{9}\ \bar{4}\ 9\ \bar{2}\ \bar{3}\ 2) + q^{2n-13}(\bar{2}\ 5\ 6\ \overline{21}\ 1\ 34\ \overline{21}\ \overline{21}\ 27\ \bar{1}\ \overline{13}\ 5\ 1\ \bar{1}) + q^{4n-13}(1\ \bar{4}\ 1\ 18\ \overline{23}\ \overline{25}\ 58\ \bar{3}\ \overline{59}\ 37\ 19\ \overline{30}\ 7\ 6\ \bar{4}\ 1) + q^{6n-9}(\bar{1}\ 0\ 9\ \bar{8}\ \overline{22}\ 31\ 18\ \overline{46}\ 7\ 29\ \overline{18}\ \bar{4}\ 7\ \bar{2}) + q^{8n-3}(2\ \bar{3}\ \bar{6}\ 11\ 5\ \overline{15}\ 1\ 9\ \bar{3}\ \bar{2}\ 1) + q^{10n+3}(\bar{1}\ 2\ 1\ \bar{4}\ 1\ 2\ \bar{1})$
10_{106}	3	10_{59}, $\{1\}su(2)_q$	$\bar{1}^3 2^2/\bar{1}^2 2\bar{1}2$	$q^{2n}(5\ \bar{4}\ 2\ \bar{1})$	$q^{-6n-23}(\bar{1}\ 2\ \bar{1}\ \bar{4}\ 8\ \bar{4}\ \overline{11}\ 20\ \bar{4}\ \overline{22}\ 20\ 6\ \overline{18}\ 6\ 7\ \bar{6}\ 0\ 2\ \bar{1}) + q^{-4n-23}(1\ \bar{2}\ 1\ 5\ \overline{10}\ 4\ 14\ \overline{26}\ 10\ 27\ \overline{40}\ 4\ 38\ \overline{28}\ \overline{12}\ 25\ \bar{6}\ \bar{9}\ 7\ 0\ \bar{2}\ 1) + q^{-2n-17}(\bar{1}\ 2\ \bar{1}\ \bar{4}\ 8\ \bar{4}\ \bar{9}\ 16\ \bar{5}\ \overline{15}\ 18\ 1\ \overline{17}\ 8\ 6\ \bar{7}\ 0\ 2\ \bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_{107}	5		$1\overline{2}3^24/\overline{1}2\overline{3}4/\overline{2}/\overline{1}^2\overline{2}3$	$q^{-2n}(\overline{2}\ 1)+(\overline{6}\ 4\ \overline{1})+q^{2n}(\overline{1}\ 1)$	$q^{-6n-13}(\overline{1}\ 2\ 1\ \overline{4}\ 1\ 2\ \overline{1})+q^{-4n-15}(1\ \overline{1}\ \overline{6}\ 8\ 9\ \overline{17}\ \overline{2}\ 14\ \overline{4}\ \overline{4}\ 2)+q^{-2n-15}(\overline{2}\ 6\ 2\ \overline{24}\ 16\ 34\ \overline{42}\ \overline{14}\ 39\ \overline{8}\ \overline{14}\ 7\ 1\ \overline{1})+q^{-15}(1\ \overline{4}\ 3\ 14\ \overline{29}\ \overline{5}\ 62\ \overline{37}\ \overline{49}\ 58\ 4\ \overline{31}\ 12\ 4\ \overline{4}\ 1)+q^{2n-11}(\overline{1}\ 0\ 7\ \overline{9}\ \overline{12}\ 32\ \overline{2}\ \overline{41}\ 24\ 20\ \overline{21}\ 0\ 6\ \overline{2})+q^{4n-5}(2\ \overline{2}\ \overline{3}\ 8\ 0\ \overline{10}\ 5\ 6\ \overline{4}\ \overline{1}\ 1)+q^{6n+1}(\overline{1}\ 1\ 0\ \overline{2}\ 1\ 1\ \overline{1})$
10_{108}	4		$\overline{1}^2\overline{2}3^2/1\overline{2}3/\overline{2}3^2$	$(2\ \overline{2}\ 1)+q^{2n}(2\ \overline{2}\ 1)$	$q^{-2n-17}(\overline{1}\ 2\ 2\ \overline{5}\ 1\ 6\ \overline{5}\ \overline{2}\ 5\ \overline{3}\ \overline{1}\ 2\ \overline{1})+q^{-17}(1\ \overline{3}\ \overline{1}\ 10\ \overline{5}\ \overline{13}\ 15\ 5\ \overline{18}\ 8\ 6\ \overline{10}\ 5\ 2\ \overline{3}\ 1)+q^{2n-15}(1\ \overline{2}\ \overline{3}\ 8\ 3\ \overline{16}\ 2\ 19\ \overline{12}\ \overline{10}\ 14\ \overline{4}\ \overline{4}\ 6\ \overline{2}\ \overline{1}\ 1)+q^{4n-13}(1\ \overline{2}\ \overline{2}\ 7\ 0\ \overline{10}\ 3\ 9\ \overline{4}\ \overline{5}\ 3\ 0\ 0\ 1\ \overline{1})+q^{6n-7}(\overline{1}\ 2\ 1\ \overline{5}\ 2\ 5\ \overline{5}\ \overline{1}\ 4\ \overline{3}\ 0\ 2\ \overline{1})$
10_{109}	3	A 10_{81}, $\{1\}su(2)_q$	$\overline{1}^22^2/\overline{1}^3\overline{2}/1^22^2$	$(\overline{5}\ 5\ \overline{2}\ 1)$	$q^{-2n-21}(\overline{1}\ 2\ \overline{1}\ \overline{6}\ 10\ 1\ \overline{23}\ 18\ 15\ \overline{34}\ 8\ 20\ \overline{21}\ 0\ 9\ \overline{6}\ \overline{1}\ 2\ \overline{1})+q^{-21}(1\ \overline{2}\ 1\ 7\ \overline{11}\ 1\ 27\ \overline{29}\ \overline{10}\ 60\ \overline{33}\ \overline{33}\ 60\ \overline{10}\ \overline{29}\ 27\ 1\ \overline{11}\ 7\ 1\ \overline{2}\ 1)+q^{2n-15}(\overline{1}\ 2\ \overline{1}\ \overline{6}\ 9\ 0\ \overline{21}\ 20\ 8\ \overline{34}\ 15\ 18\ \overline{23}\ 1\ 10\ \overline{6}\ \overline{1}\ 2\ \overline{1})$
10_{110}	5		$12\overline{3}\overline{4}/2^33\overline{4}/1\overline{2}\overline{3}$	$(1\ \overline{1})+q^{2n}(5\ \overline{4}\ 1)+q^{4n}(1\ \overline{1})$	$q^{-2n-11}(\overline{1}\ 1\ 1\ \overline{2}\ 0\ 1\ \overline{1})+q^{-13}(1\ 0\ \overline{6}\ 2\ 9\ \overline{8}\ \overline{5}\ 8\ \overline{2}\ \overline{3}\ 2)+q^{2n-13}(\overline{2}\ 5\ 6\ \overline{19}\ \overline{2}\ 30\ \overline{14}\ \overline{21}\ 21\ 0\ \overline{11}\ 5\ 1\ \overline{1})+q^{4n-13}(1\ \overline{4}\ 1\ 17\ \overline{19}\ \overline{25}\ 47\ 5\ \overline{50}\ 25\ 17\ \overline{24}\ 7\ 5\ \overline{4}\ 1)+q^{6n-9}(\overline{1}\ 0\ 8\ \overline{5}\ \overline{20}\ 21\ 20\ \overline{34}\ 0\ 23\ \overline{13}\ \overline{3}\ 6\ \overline{2})+q^{8n-3}(2\ \overline{2}\ \overline{5}\ 6\ 5\ \overline{8}\ \overline{1}\ 5\ \overline{2}\ \overline{1}\ 1)+q^{10n+3}(\overline{1}\ 1\ 1\ \overline{2}\ 0\ 1\ \overline{1})$
10_{111}	4		$1^22^2\overline{3}/2^2/\overline{1}2\overline{3}/2$	$q^{2n}(1)+q^{4n}(\overline{3}\ 3\ \overline{1})+q^{6n}(\overline{3}\ 2\ \overline{1})$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-11}(\overline{1}\ 2\ 1\ \overline{6}\ 4\ 6\ \overline{8}\ 1\ 6\ \overline{4}\ 0\ 2\ \overline{1})+q^{8n-9}(\overline{4}\ 2\ 11\ \overline{14}\ \overline{10}\ 26\ \overline{7}\ \overline{21}\ 20\ 1\ \overline{13}\ 8\ 1\ \overline{3}\ 1)+q^{10n-11}(1\ 0\ \overline{6}\ 3\ 16\ \overline{13}\ \overline{22}\ 29\ 9\ \overline{34}\ 12\ 14\ \overline{18}\ 4\ 4\ \overline{4}\ 1)+q^{12n-7}(1\ \overline{3}\ \overline{1}\ 12\ \overline{4}\ \overline{19}\ 16\ 15\ \overline{22}\ 2\ 14\ \overline{10}\ 1\ 4\ \overline{3}\ 1)+q^{14n-1}(\overline{1}\ 2\ 1\ \overline{6}\ 1\ 7\ \overline{5}\ \overline{3}\ 5\ \overline{2}\ \overline{1}\ 2\ \overline{1})$
10_{112}	3	$\{2\}su(2)_q$ DADW	$1^4\overline{2}/1\overline{2}^2/\overline{1}^22^2$	$q^{2n}(6\ \overline{4}\ 3\ \overline{1})$	$q^{2n-19}(\overline{1}\ 3\ 0\ \overline{10}\ 9\ 11\ \overline{22}\ 1\ 24\ \overline{18}\ \overline{7}\ 20\ \overline{11}\ \overline{5}\ 11\ \overline{6}\ \overline{1}\ 3\ \overline{1})+q^{4n-19}(1\ \overline{3}\ 0\ 10\ \overline{13}\ \overline{8}\ 34\ \overline{15}\ \overline{36}\ 48\ 7\ \overline{49}\ 34\ 14\ \overline{32}\ 19\ 5\ \overline{14}\ 7\ 1\ \overline{3}\ 1)+q^{6n-13}(\overline{1}\ 3\ 0\ \overline{9}\ 10\ 8\ \overline{24}\ 6\ 25\ \overline{27}\ \overline{6}\ 24\ \overline{14}\ \overline{5}\ 11\ \overline{6}\ \overline{1}\ 3\ \overline{1})$
10_{113}	4		$\overline{1}2\overline{3}/2\overline{3}/1^32\overline{3}/2$	$q^{2n}(5\ \overline{3}\ 1)+q^{4n}(5\ \overline{4}\ 1)$	$q^{2n-13}(\overline{1}\ 3\ 0\ \overline{9}\ 8\ 8\ \overline{15}\ 3\ 10\ \overline{7}\ \overline{1}\ 3\ \overline{1})+q^{4n-11}(\overline{2}\ 8\ 2\ \overline{28}\ 17\ 36\ \overline{44}\ \overline{10}\ 42\ \overline{15}\ \overline{15}\ 13\ 0\ \overline{3}\ 1)+q^{6n-13}(1\ \overline{1}\ \overline{7}\ 14\ 11\ \overline{51}\ 14\ 71\ \overline{64}\ \overline{36}\ 70\ \overline{15}\ \overline{30}\ 19\ 2\ \overline{5}\ 1)+q^{8n-9}(1\ \overline{5}\ 2\ 20\ \overline{29}\ \overline{19}\ 70\ \overline{21}\ \overline{62}\ 56\ 12\ \overline{36}\ 13\ 6\ \overline{5}\ 1)+q^{10n-3}(\overline{1}\ 4\ \overline{1}\ \overline{13}\ 15\ 10\ \overline{27}\ 6\ 17\ \overline{12}\ \overline{2}\ 4\ \overline{1})$
10_{114}	4		$\overline{1}\overline{2}3/\overline{2}3/\overline{2}^2\overline{3}/1^223^2$	$(\overline{4}\ 3\ \overline{1})+q^{2n}(\overline{3}\ 3\ \overline{1})$	$q^{-2n-15}(\overline{1}\ 3\ \overline{1}\ \overline{8}\ 11\ 4\ \overline{17}\ 7\ 9\ \overline{9}\ 0\ 3\ \overline{1})+q^{-13}(\overline{1}\ 4\ \overline{2}\ \overline{14}\ 19\ 12\ \overline{36}\ 7\ 29\ \overline{18}\ \overline{9}\ 12\ \overline{1}\ \overline{3}\ 1)+q^{2n-15}(1\ \overline{2}\ \overline{4}\ 12\ \overline{1}\ \overline{28}\ 29\ 23\ \overline{53}\ 8\ 41\ \overline{24}\ \overline{12}\ 14\ \overline{1}\ \overline{3}\ 1)+q^{4n-11}(1\ \overline{4}\ 2\ 12\ \overline{21}\ \overline{1}\ 38\ \overline{31}\ \overline{20}\ 43\ \overline{8}\ \overline{21}\ 13\ 2\ \overline{4}\ 1)+q^{6n-5}(\overline{1}\ 3\ 0\ \overline{8}\ 9\ 3\ \overline{15}\ 7\ 8\ \overline{9}\ 0\ 3\ \overline{1})$
10_{115}	5	A	$\overline{1}23^24/2\overline{3}4/\overline{1}\overline{2}3/\overline{1}\overline{2}^2$	$q^{-2n}(\overline{2}\ 1)+(\overline{7}\ 5\ \overline{1})+q^{2n}(\overline{2}\ 1)$	$q^{-6n-13}(\overline{1}\ 2\ 1\ \overline{4}\ 1\ 2\ \overline{1})+q^{-4n-15}(1\ \overline{2}\ \overline{6}\ 11\ 8\ \overline{20}\ 0\ 15\ \overline{5}\ \overline{4}\ 2)+q^{-2n-15}(\overline{2}\ 8\ 0\ \overline{31}\ 28\ 39\ \overline{62}\ \overline{10}\ 51\ \overline{15}\ \overline{16}\ 9\ 1\ \overline{1})+q^{-15}(1\ \overline{5}\ 5\ 17\ \overline{42}\ 1\ 86\ \overline{61}\ \overline{61}\ 86\ 1\ \overline{42}\ 17\ 5\ \overline{5}\ 1)+q^{2n-11}(\overline{1}\ 1\ 9\ \overline{16}\ \overline{15}\ 51\ \overline{10}\ \overline{62}\ 39\ 28\ \overline{31}\ 0\ 8\ \overline{2})+q^{4n-5}(2\ \overline{4}\ \overline{5}\ 15\ 0\ \overline{20}\ 8\ 11\ \overline{6}\ \overline{2}\ 1)+q^{6n+1}(\overline{1}\ 2\ 1\ \overline{4}\ 1\ 2\ \overline{1})$
10_{116}	3		$1^2\overline{2}^2/\overline{1}^22^2/1^3\overline{2}$	$q^{2n}(6\ \overline{5}\ 3\ \overline{1})$	$q^{2n-19}(\overline{1}\ 3\ 0\ \overline{11}\ 10\ 13\ \overline{27}\ 2\ 30\ \overline{24}\ \overline{9}\ 27\ \overline{14}\ \overline{7}\ 14\ \overline{6}\ \overline{2}\ 3\ \overline{1})+q^{4n-19}(1\ \overline{3}\ 0\ 11\ \overline{14}\ \overline{10}\ 40\ \overline{18}\ \overline{45}\ 59\ 7\ \overline{64}\ 42\ 17\ \overline{43}\ 22\ 8\ \overline{17}\ 7\ 2\ \overline{3}\ 1)+q^{6n-13}(\overline{1}\ 3\ 0\ \overline{10}\ 11\ 10\ \overline{29}\ 8\ 32\ \overline{33}\ \overline{7}\ 32\ \overline{17}\ \overline{7}\ 14\ \overline{6}\ \overline{2}\ 3\ \overline{1})$
10_{117}	4		$1^22\overline{3}/2^2\overline{3}/\overline{1}2\overline{3}/2$	$q^{2n}(5\ \overline{3}\ 1)+q^{4n}(5\ \overline{3}\ 1)$	$q^{2n-13}(\overline{1}\ 3\ 0\ \overline{9}\ 8\ 8\ \overline{15}\ 3\ 10\ \overline{7}\ \overline{1}\ 3\ \overline{1})+q^{4n-11}(\overline{2}\ 7\ 3\ \overline{26}\ 13\ 36\ \overline{39}\ \overline{13}\ 40\ \overline{12}\ \overline{15}\ 12\ 0\ \overline{3}\ 1)+q^{6n-13}(1\ \overline{1}\ \overline{6}\ 12\ 11\ \overline{42}\ 8\ 65\ \overline{49}\ \overline{37}\ 60\ \overline{7}\ \overline{27}\ 15\ 2\ \overline{4}\ 1)+q^{8n-9}(1\ \overline{4}\ 1\ 16\ \overline{22}\ \overline{19}\ 56\ \overline{12}\ \overline{55}\ 42\ 14\ \overline{29}\ 9\ 5\ \overline{4}\ 1)+q^{10n-3}(\overline{1}\ 3\ 0\ \overline{10}\ 10\ 10\ \overline{21}\ 2\ 14\ \overline{8}\ \overline{2}\ 3\ \overline{1})$
10_{118}	3	A	$1^3\overline{2}/1\overline{2}^3/\overline{1}^22^2$	$(\overline{6}\ 5\ \overline{3}\ 1)$	$q^{-2n-21}(\overline{1}\ 3\ \overline{1}\ \overline{8}\ 14\ 2\ \overline{28}\ 22\ 19\ \overline{40}\ 11\ 25\ \overline{25}\ 1\ 13\ \overline{8}\ \overline{1}\ 3\ \overline{1})+q^{-21}(1\ \overline{3}\ 1\ 9\ \overline{16}\ 0\ 33\ \overline{38}\ \overline{15}\ 70\ \overline{42}\ \overline{42}\ 70\ \overline{15}\ \overline{38}\ 33\ 0\ \overline{16}\ 9\ 1\ \overline{3}\ 1)+q^{2n-15}(\overline{1}\ 3\ \overline{1}\ \overline{8}\ 13\ 1\ \overline{25}\ 25\ 11\ \overline{40}\ 19\ 22\ \overline{28}\ 2\ 14\ \overline{8}\ \overline{1}\ 3\ \overline{1})$
10_{119}	4		$1^22\overline{3}^2/2\overline{3}/\overline{1}2\overline{3}/2$	$(\overline{5}\ 3\ \overline{1})+q^{2n}(\overline{4}\ 3\ \overline{1})$	$q^{-2n-15}(\overline{1}\ 3\ \overline{2}\ \overline{7}\ 13\ 1\ \overline{19}\ 10\ 9\ \overline{10}\ 0\ 3\ \overline{1})+q^{-13}(\overline{2}\ 6\ \overline{3}\ \overline{18}\ 28\ 12\ \overline{49}\ 14\ 36\ \overline{25}\ \overline{10}\ 14\ \overline{1}\ \overline{3}\ 1)+q^{2n-15}(1\ \overline{1}\ \overline{5}\ 12\ 1\ \overline{34}\ 33\ 31\ \overline{68}\ 6\ 55\ \overline{29}\ \overline{18}\ 18\ 0\ \overline{4}\ 1)+q^{4n-11}(1\ \overline{4}\ 2\ 12\ \overline{23}\ 0\ 43\ \overline{37}\ \overline{25}\ 50\ \overline{7}\ \overline{25}\ 13\ 3\ \overline{4}\ 1)+q^{6n-5}(\overline{1}\ 3\ \overline{1}\ \overline{7}\ 11\ 1\ \overline{16}\ 10\ 8\ \overline{10}\ 0\ 3\ \overline{1})$
10_{120}	5		$23^2\overline{4}/1\overline{2}3/1\overline{2}34^2/3^2$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\overline{5}4)+q^{8n}(\overline{5}\ 3)+q^{10n}(\overline{1})$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-3}(1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{8n-5}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{10n-5}(4\ \overline{2}\ \overline{11}\ 17\ 8\ \overline{27}\ 8\ 17\ \overline{11}\ \overline{2}\ 4)+q^{12n-5}(3\ 7\ \overline{11}\ \overline{21}\ 38\ 20\ \overline{60}\ 8\ 44\ \overline{21}\ \overline{11}\ 10)+q^{14n-3}(2\ 11\ \overline{11}\ \overline{38}\ 43\ 46\ \overline{79}\ \overline{10}\ 62\ \overline{19}\ \overline{19}\ 10)+q^{16n+1}(3\ \overline{1}\ \overline{29}\ 11\ 55\ \overline{46}\ \overline{40}\ 52\ 0\ \overline{21}\ 6)+q^{18n+7}(\overline{6}\ \overline{2}\ 23\ \overline{4}\ \overline{27}\ 14\ 11\ \overline{9})+q^{20n+13}(4\ 1\ \overline{5}\ 1\ 3)+q^{22n+19}(\overline{1})$
10_{121}	4		$\overline{1}2^23^2/\overline{2}^2\overline{3}/1^22\overline{3}/2$	$q^{2n}(6\ \overline{4}\ 1)+q^{4n}(5\ \overline{3}\ 1)$	$q^{2n-13}(\overline{1}\ 4\ \overline{1}\ \overline{13}\ 14\ 11\ \overline{25}\ 5\ 16\ \overline{11}\ \overline{2}\ 4\ \overline{1})+q^{4n-11}(\overline{3}\ 9\ 7\ \overline{38}\ 14\ 58\ \overline{56}\ \overline{26}\ 62\ \overline{15}\ \overline{25}\ 17\ 1\ \overline{4}\ 1)+q^{6n-13}(1\ \overline{1}\ \overline{7}\ 12\ 16\ \overline{48}\ \overline{4}\ 86\ \overline{46}\ \overline{65}\ 73\ 5\ \overline{40}\ 16\ 5\ \overline{5}\ 1)+q^{8n-9}(1\ \overline{4}\ 1\ 17\ \overline{21}\ \overline{24}\ 59\ \overline{2}\ \overline{65}\ 38\ 24\ \overline{31}\ 6\ 6\ \overline{4}\ 1)+q^{10n-3}(\overline{1}\ 3\ 0\ \overline{10}\ 9\ 11\ \overline{20}\ 0\ 14\ \overline{7}\ \overline{2}\ 3\ \overline{1})$
10_{122}	4		$1\overline{2}^2\overline{3}^2/2^23^2/\overline{1}2\overline{3}/\overline{1}2^2$	$(\overline{4}\ 3\ \overline{1})+q^{2n}(\overline{4}\ 4\ \overline{1})$	$q^{-2n-15}(\overline{1}\ 3\ \overline{1}\ \overline{7}\ 10\ 2\ \overline{15}\ 8\ 8\ \overline{9}\ 0\ 3\ \overline{1})+q^{-13}(\overline{1}\ 5\ \overline{5}\ \overline{12}\ 26\ 2\ \overline{39}\ 18\ 27\ \overline{23}\ \overline{8}\ 13\ \overline{1}\ \overline{3}\ 1)+q^{2n-15}(1\ \overline{2}\ \overline{5}\ 15\ \overline{4}\ \overline{35}\ 46\ 19\ \overline{75}\ 25\ 51\ \overline{38}\ \overline{13}\ 19\ \overline{1}\ \overline{4}\ 1)+q^{4n-11}(1\ \overline{5}\ 4\ 15\ \overline{30}\ 6\ 51\ \overline{51}\ \overline{20}\ 63\ \overline{15}\ \overline{28}\ 18\ 3\ \overline{5}\ 1)+q^{6n-5}(\overline{1}\ 4\ \overline{2}\ \overline{10}\ 15\ \overline{1}\ \overline{22}\ 14\ 10\ \overline{14}\ 0\ 4\ \overline{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_{123}	3	A	$\bar{1}^2 2^2/1^3 2^2/\bar{1}^2\bar{2}^3$	$(\bar{8}\,6\,\bar{4}\,1)$	$q^{-2n-21}(\bar{1}\,4\,\bar{2}\,\overline{12}\,22\,2\,\overline{45}\,36\,29\,\overline{66}\,19\,40\,\overline{41}\,1\,21\,\overline{12}\,\bar{2}\,4\,\bar{1}) + q^{-21}(1\,\bar{4}\,2\,13\,\overline{25}\,1\,53\,\overline{61}\,\overline{24}\,113\,\overline{67}\,\overline{67}\,113\,\overline{24}\,\overline{61}\,53\,1\,\overline{25}\,13\,2\,\bar{4}\,1) + q^{2n-15}(\bar{1}\,4\,\bar{2}\,\overline{12}\,21\,1\,\overline{41}\,40\,19\,\overline{66}\,29\,36\,\overline{45}\,2\,22\,\overline{12}\,\bar{2}\,4\,\bar{1})$
10_{124}	3		$12^5/12^3$	$q^{2n}(1) + q^{4n}(0\,1) + q^{6n}(1\,0\,1) + q^{8n}(0\,1\,0\,1) + q^{10n}(0\,\bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1001) + q^{6n-3}(1001001) + q^{8n-5}(1001001001) + q^{10n-7}(1001001001001) + q^{12n-9}(1001001001001001) + q^{14n-11}(1001001001001001001) + q^{16n-13}(1001001001001001001001) + q^{18n-7}(\bar{1}\bar{1}\bar{1}\bar{1}\bar{2}\bar{2}\bar{3}\bar{2}\bar{3}\bar{3}\bar{3}\bar{3}\bar{4}\bar{2}\bar{3}\bar{2}\bar{2}\bar{1}\bar{1}) + q^{20n+1}(11112322322211) + q^{22n+11}(\bar{1}00\bar{1}\bar{1}0\bar{1})$
10_{125}	3	FA, $\{1\}su(n)_q$	$\overline{12}^3/\bar{1}2^5$	$(1\,0\,1)$	$q^{-2n-17}(\bar{1}00\bar{1}\bar{1}0\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}0\bar{1}) + q^{-17}(1002212332222101) + q^{2n-11}(\bar{1}00\bar{1}\bar{2}\bar{1}0\bar{1}\bar{2}000\bar{1})$
10_{126}	3		$1\bar{2}^3/12^5$	$q^{2n}(1) + q^{4n}(2\,0\,1)$	$q^{2n+1}(1) + q^{4n-1}(1001) + q^{6n-7}(1\bar{1}\bar{1}21\bar{3}11\bar{1}00\bar{1}) + q^{8n-9}(1012025\bar{1}16113101) + q^{10n-3}(\bar{1}0\bar{1}\bar{2}\bar{1}\bar{1}\bar{4}\bar{1}\bar{1}\bar{2}\bar{1}0\bar{1})$
10_{127}	3		$1^5 2/\bar{1}^2 2^2$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{2}1\bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1001) + q^{6n-3}(1001001) + q^{8n-5}(1001001001) + q^{10n-9}(\bar{1}\bar{2}20\bar{6}\bar{1}3\bar{6}\bar{3}1\bar{4}\bar{2}1\bar{2}) + q^{12n-7}(1\bar{1}15\bar{3}\bar{2}100\bar{5}81\bar{3}40\bar{1}1) + q^{14n-1}(\bar{1}10\bar{3}12\bar{4}02\bar{2}01\bar{1})$
10_{128}	4		$1^2 2^2 3/2^2 3/\bar{1}23$	$q^{2n}(1) + q^{4n}(0\,1) + q^{6n}(1\,0\,1) + q^{8n}(0\,0\,1) + q^{10n}(\bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1001) + q^{6n-3}(1001001) + q^{8n-5}(1001001001) + q^{10n-7}(1001001001001) + q^{12n-9}(1001001001001001) + q^{14n-9}(11\bar{1}0100010\bar{1}010\bar{1}01) + q^{16n-5}(1\bar{1}\bar{2}00\bar{1}\bar{1}0\bar{1}\bar{2}000\bar{1}\bar{1}1) + q^{18n+1}(\bar{1}\bar{1}01\bar{1}00\bar{2}\bar{1}1\bar{1}\bar{1}\bar{1}) + q^{20n+9}(11100111) + q^{22n+19}(\bar{1})$
10_{129}	4	8_8, $\{1\}su(n)_q$	$\bar{1}^2\bar{2}^2 3/2^2 3/1\bar{2}3$	$(\bar{1}\,1) + q^{2n}(\bar{1}\,1)$	$q^{-2n-15}(1\,\bar{3}\,2\,5\,\bar{8}\,0\,9\,\bar{7}\,\bar{2}\,5\,\bar{2}\,\bar{1}\,1) + q^{-15}(\bar{1}\,4\,\bar{4}\,\bar{7}\,18\,\bar{4}\,\overline{22}\,23\,5\,\overline{22}\,12\,6\,\bar{9}\,2\,2\,\bar{1}) + q^{2n-11}(1\,0\,\bar{6}\,8\,9\,\overline{20}\,4\,20\,\overline{15}\,\bar{2}\,11\,\bar{4}\,\bar{1}\,2) + q^{4n-1}(2\bar{2}\bar{3}40\bar{3}10\bar{1}) + q^{6n+1}(\bar{1}10\bar{2}11\bar{1})$
10_{130}	4		$\bar{1}^3 2\bar{3}/1^2 23^2/2$	$q^{2n}(0\,1) + q^{4n}(0\,1)$	$q^{2n-1}(\bar{1}110\bar{1}) + q^{4n-5}(11\bar{1}\bar{1}220\bar{1}01) + q^{6n-5}(1111\bar{1}131\bar{1}01) + q^{8n-1}(1\bar{1}00\bar{2}12\bar{1}\bar{1}1) + q^{10n+5}(\bar{1}0\bar{1}\bar{1}00\bar{1})$
10_{131}	4		$1^3 2\bar{3}/\bar{1}^2 23^2/2$	$q^{2n}(1)+q^{4n}(2\bar{1})+q^{6n}(1\bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1001) + q^{6n-5}(\bar{1}\bar{1}40\bar{5}22\bar{2}) + q^{8n-5}(\bar{1}\bar{2}04\bar{1}\bar{7}24\bar{3}\bar{1}1) + q^{10n-3}(1\bar{1}\bar{2}61\bar{8}25\bar{3}\bar{1}1) + q^{12n+1}(1\bar{2}15\bar{5}\bar{2}6\bar{1}\bar{2}1) + q^{14n+7}(\bar{1}10\bar{2}11\bar{1})$
10_{132}	4	5_1, $\{1\}su(n)_q$	$\bar{1}23/1^3\bar{2}^3 3/2$	$q^{2n}(1) + q^{4n}(0\,1)$	$q^{2n+1}(1) + q^{4n-5}(10\bar{1}11) + q^{6n-3}(11\bar{1}02) + q^{8n-1}(1000011001) + q^{10n+5}(\bar{1}0\bar{1}\bar{1}00\bar{1})$
10_{133}	4		$1^2\overline{23}/12^3 3/2^2\bar{3}^2$	$q^{2n}(1) + q^{4n}(1) + q^{6n}(1\,\bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1001) + q^{6n-1}(11\bar{1}01) + q^{8n-5}(\bar{1}\bar{1}20\bar{2}) + q^{10n-1}(\bar{1}03\bar{3}\bar{3}20\bar{2}) + q^{12n+1}(1\bar{1}13\bar{3}04\bar{1}\bar{1}1) + q^{14n+7}(\bar{1}10\bar{2}11\bar{1})$
10_{134}	4		$12^2 3/123^3/\bar{1}2^2\bar{3}$	$q^{2n}(1) + q^{4n}(0\,1) + q^{6n}(1\,0\,1) + q^{8n}(1\,\bar{1}\,1) + q^{10n}(\bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1001) + q^{6n-3}(1001001) + q^{8n-5}(1001001001) + q^{10n-7}(1001001001001) + q^{12n-9}(1001001001001001) + q^{14n-9}(11\bar{2}03\bar{2}\bar{2}20\bar{2}1\bar{1}\bar{1}1\bar{1}\bar{1}1) + q^{16n-5}(1\bar{2}\bar{3}30\bar{6}13\bar{5}\bar{1}2\bar{3}10\bar{2}1) + q^{18n+1}(\bar{1}020\bar{2}31\bar{3}11\bar{1}0\bar{1}) + q^{20n+9}(10010011) + q^{22n+19}(\bar{1})$
10_{135}	4		$12^2 3/1^2\overline{23}^3/\bar{2}$	$(\bar{2}\,2) + q^{2n}(\bar{1}\,1)$	$q^{-2n-11}(\bar{1}\bar{2}40\bar{6}22\bar{3}) + q^{-11}(1\,1\,\bar{5}\,6\,9\,\overline{13}\,0\,13\,\bar{4}\,\bar{3}\,3) + q^{2n-9}(11\bar{4}29\bar{9}\bar{5}120\bar{5}21) + q^{4n-5}(1\bar{2}\bar{1}5\bar{3}\bar{5}51\bar{3}) + q^{6n+1}(\bar{1}10\bar{2}11\bar{1})$
10_{136}	4		$12^2\bar{3}/\bar{1}2\bar{3}/\overline{12}3^2$	$(1\,\bar{1}) + q^{2n}(1)$	$q^{-2n-11}(\bar{1}11\bar{2}01\bar{1}) + q^{-11}(1\bar{1}\bar{1}30\bar{2}20\bar{1}1) + q^{2n-5}(\bar{2}\bar{1}30\bar{4}10\bar{1}) + q^{4n+1}(13\bar{1}\bar{1}2) + q^{6n+9}(\bar{1})$
10_{137}	5	10_{155}, $\{1\}su(2)_q$	$1^2\bar{2}^2\bar{3}/\bar{1}23^2\bar{4}^2/23^2 4/\bar{2}$	$(\bar{1}) + q^{2n}(\bar{2}\,1) + q^{4n}(\bar{1})$	$q^{-2n-3}(\bar{1})+q^{-7}(10\bar{1}\bar{1}01)+q^{2n-7}(\bar{1}13\bar{5}\bar{4}41\bar{2})+q^{4n-1}(3\bar{1}\bar{6}35\bar{2}\bar{1}1)+q^{6n-1}(\bar{1}12\bar{5}\bar{2}40\bar{2})+q^{8n+5}(2\bar{1}\bar{2}21) + q^{10n+11}(\bar{1})$
10_{138}	5		$\bar{1}234/2\overline{34}/23\bar{4}/1^2\bar{2}^2$	$(0\,\bar{1}) + q^{2n}(2\,\bar{2}\,1) + q^{4n}(\bar{1})$	$q^{-2n-11}(\bar{1}00\bar{1}\bar{1}0\bar{1}) + q^{-13}(12\bar{2}\bar{1}40\bar{3}20\bar{1}2) + q^{2n-13}(\bar{2}14\bar{4}\bar{6}52\bar{7}01\bar{3}20\bar{1}) + q^{4n-13}(1\bar{2}\bar{1}7\bar{1}\bar{9}68\bar{8}05\bar{3}31\bar{2}1) + q^{6n-7}(\bar{1}12\bar{4}\bar{2}6\bar{3}\bar{5}3\bar{1}\bar{1}1\bar{1}) + q^{8n+1}(1\bar{1}02\bar{1}\bar{1}11) + q^{10n+11}(\bar{1})$
10_{139}	3		$1^3 2^3/1^3 2$	$q^{2n}(1) + q^{4n}(0\,1) + q^{6n}(1\,0\,1) + q^{8n}(0\,1\,0\,1) + q^{10n}(1\,\bar{1})$	$q^{2n+1}(1) + q^{4n-1}(1001) + q^{6n-3}(1001001) + q^{8n-5}(1001001001) + q^{10n-7}(1001001001001) + q^{12n-9}(1001001001001001) + q^{14n-11}(1001001001001001001) + q^{16n-13}(1001001001001001001001) + q^{18n-7}(\bar{1}\bar{1}\bar{1}0\bar{1}\bar{2}\bar{2}0\bar{2}\bar{3}\bar{1}\bar{2}\bar{3}\bar{1}\bar{3}\bar{1}\bar{1}\bar{1}\bar{1}) + q^{20n+1}(110\bar{1}13\bar{1}0301101) + q^{22n+11}(\bar{1}11\bar{2}01\bar{1})$
10_{140}	4		$1\overline{23}^3/123^3/2$	$q^{4n}(0\,1)$	$q^{4n-5}(11\bar{1}12) + q^{6n+1}(\bar{1}\bar{2}10\bar{1}\bar{1}) + q^{8n-1}(1011022001) + q^{10n+5}(\bar{1}0\bar{1}\bar{1}00\bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_{141}	3		$12^3/1^2\bar{2}^4$	$q^{2n}(\bar{1}\ 1\ \bar{1})$	$q^{2n-11}(\bar{1}\ 0\ \bar{1}\ 0\ 1\ \bar{3}\ 0\ 3\ \bar{1}\ \bar{2}\ 1\ 1\ \bar{1})+q^{4n-11}(1\ \bar{1}\ 0\ 2\ \bar{1}\ 2\ 0\ \bar{3}\ 3\ 2\ \bar{4}\ \bar{1}\ 3\ \bar{1}\ \bar{1}\ 1)+q^{6n-5}(\bar{1}\ 1\ 0\ \bar{1}\ 1\ \bar{1}\ \bar{1}\ 2\ 0\ \bar{2}\ 1\ 1\ \bar{1})$
10_{142}	4		$\bar{1}23^3/1^223^3$	$q^{2n}(1)+q^{4n}(0\ 1)+q^{6n}(1\ 0\ 1)+q^{8n}(1\ 0\ 1)+q^{10n}(\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-3}(1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{8n-5}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{10n-7}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{12n-9}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{14n-9}(1\ 1\ \bar{1}\ 0\ 2\ 0\ \bar{1}\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1)+q^{16n-5}(1\ \bar{1}\ \bar{2}\ 1\ 1\ \bar{2}\ 0\ 2\ \bar{1}\ \bar{1}\ 1\ \bar{1}\ 0\ 0\ \bar{1}\ 1)+q^{18n+1}(\bar{1}\ \bar{1}\ 0\ 0\ \bar{3}\ \bar{1}\ \bar{1}\ 3\ \bar{2}\ 0\ \bar{2}\ \bar{1}\ \bar{1})+q^{20n+9}(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)+q^{22n+19}(\bar{1})$
10_{143}	3		$1\bar{2}^3/1^32^3$	$q^{2n}(1)+q^{4n}(2\ \bar{1}\ 1)$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-7}(2\ \bar{1}\ \bar{2}\ 5\ 1\ \bar{5}\ 3\ 2\ \bar{3}\ 1\ 1\ \bar{1})+q^{8n-9}(1\ \bar{1}\ 0\ 2\ \bar{4}\ 1\ 6\ \bar{7}\ \bar{1}\ 8\ \bar{3}\ \bar{2}\ 4\ 0\ \bar{1}\ 1)+q^{10n-3}(\bar{1}\ 1\ 0\ \bar{2}\ 2\ 0\ \bar{4}\ 2\ 1\ \bar{3}\ 0\ 1\ \bar{1})$
10_{144}	4		$\bar{1}2\bar{3}^2/1\bar{2}3/1^22^2$	$q^{2n}(2\ \bar{2})+q^{4n}(2\ \bar{1})$	$q^{2n-9}(\bar{1}\ \bar{3}\ 2\ 0\ \bar{5}\ 1\ 1\ \bar{3})+q^{4n-9}(1\ 2\ \bar{6}\ 2\ 13\ \bar{8}\ \bar{5}\ 12\ \bar{2}\ \bar{4}\ 3)+q^{6n-7}(1\ 1\ \bar{6}\ \bar{1}\ 13\ \bar{7}\ \overline{12}\ 10\ 2\ \bar{7}\ 1\ 1)+q^{8n-3}(1\ \bar{3}\ \bar{2}\ 9\ \bar{2}\ 9\ 6\ 3\ \bar{3})+q^{10n+3}(\bar{1}\ 2\ 1\ \bar{4}\ 1\ 2\ \bar{1})$
10_{145}	4		$1^22\bar{3}/\bar{1}23/12^23$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(1)+q^{8n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-3}(1001001)+q^{8n-5}(1001001001)+q^{10n-1}(2\ 0\ \bar{2}\ 3\ 0\ \bar{2}\ 1)+q^{12n+1}(1\ 2\ \bar{2}\ 1\ 3\ \bar{2}\ 0\ 1)+q^{14n+5}(\bar{1}\ 0\ 1\ \bar{1}\ \bar{1})+q^{16n+9}(\bar{1}\ 2\ 0\ \bar{2}\ 1)+q^{18n+15}(1\ \bar{1})$
10_{146}	4		$1^2\overline{23}^2/2^2/1\bar{2}3/\bar{2}$	$(\bar{2}\ 1)+q^{2n}(\bar{2}\ 1)$	$q^{-2n-7}(2\ 0\ \bar{2}\ 1\ 1\ \bar{1})+q^{-9}(\bar{1}\ \bar{1}\ 5\ 1\ \bar{7}\ 2\ 5\ \bar{2}\ \bar{1}\ 1)+q^{2n-9}(1\ \bar{1}\ \bar{4}\ 6\ 4\ \overline{11}\ 1\ 8\ \bar{3}\ \bar{2}\ 1)+q^{4n-5}(1\ \bar{3}\ 0\ 7\ \bar{5}\ \bar{5}\ 7\ 0\ \bar{3}\ 1)+q^{6n+1}(\bar{1}\ 2\ 1\ \bar{3}\ 1\ 2\ \bar{1})$
10_{147}	4		$1\overline{23}/\bar{1}2\bar{3}/\bar{1}2^33$	$(1\ \bar{1})+q^{2n}(2\ \bar{1})$	$q^{-2n-11}(\bar{1}\ 1\ 1\ \bar{2}\ 0\ 1\ \bar{1})+q^{-11}(1\ \bar{2}\ \bar{1}\ 5\ \bar{2}\ \bar{4}\ 4\ 0\ \bar{2}\ 1)+q^{2n-9}(1\ \bar{1}\ \bar{4}\ 3\ 4\ \bar{5}\ \bar{1}\ 4\ \bar{2}\ \bar{1}\ 1)+q^{4n-7}(1\ \bar{1}\ \bar{3}\ 3\ 4\ \bar{3}\ \bar{2}\ 2\ 0\ \bar{1})+q^{6n-1}(\bar{1}\ 1\ 2\ \bar{2}\ \bar{1}\ 1)$
10_{148}	3		$1^22^3/\bar{1}^22\bar{1}2$	$q^{2n}(1)+q^{4n}(3\ \bar{1}\ 1)$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-7}(2\ \bar{2}\ \bar{3}\ 7\ 1\ \bar{8}\ 5\ 3\ \bar{4}\ 1\ 1\ \bar{1})+q^{8n-9}(1\ \bar{1}\ 1\ 3\ \bar{5}\ 3\ 10\ \overline{10}\ \bar{1}\ 12\ \bar{4}\ \bar{2}\ 5\ 0\ \bar{1}\ 1)+q^{10n-3}(\bar{1}\ 1\ \bar{1}\ \bar{3}\ 3\ \bar{1}\ \bar{7}\ 3\ 1\ \bar{4}\ 0\ 1\ \bar{1})$
10_{149}	3		$\bar{1}^22^3/1^32^2$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{3}2\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-3}(1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{8n-5}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{10n-9}(\bar{1}\ \bar{2}\ 4\ 1\ \overline{10}\ 3\ 8\ \overline{12}\ \bar{1}\ 7\ \bar{7}\ \bar{2}\ 3\ \bar{2})+q^{12n-7}(1\ \bar{2}\ 1\ 8\ \bar{9}\ \bar{6}\ 20\ \bar{6}\ \overline{14}\ 17\ \bar{1}\ \bar{9}\ 7\ 0\ \bar{2}\ 1)+q^{14n-1}(\bar{1}\ 2\ 0\ \bar{6}\ 5\ 5\ \overline{10}\ 2\ 6\ \bar{5}\ 0\ 2\ \bar{1})$
10_{150}	4		$\bar{2}3/1^2\overline{23}/12^23^2$	$q^{2n}(1)+q^{4n}(\bar{2}\ 2\ \bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1001)+q^{6n-11}(\bar{1}\ 1\ 1\ \bar{3}\ 1\ 3\ \bar{3}\ 0\ 2\ \bar{2}\ 0\ 1\ \bar{1})+q^{8n-11}(1\ \bar{2}\ \bar{2}\ 6\ \bar{1}\ \bar{8}\ 6\ 4\ \bar{8}\ 3\ 3\ \bar{5}\ 3\ 1\ \bar{2}\ 1)+q^{10n-5}(\bar{1}\ 2\ 3\ \bar{5}\ \bar{2}\ 8\ \bar{2}\ \bar{6}\ 5\ \bar{1}\ 2\ \bar{2}\ \bar{1})+q^{12n+5}(\bar{1}\ 0\ 2\ \bar{1}\ \bar{1}\ 1)$
10_{151}	4		$1^22^23/1\bar{2}^2\bar{3}/2\bar{3}$	$q^{2n}(4\ \bar{2}\ 1)+q^{4n}(1)$	$q^{2n-13}(\bar{1}\ 2\ 1\ \bar{6}\ 3\ 6\ \bar{8}\ 1\ 6\ \bar{4}\ \bar{1}\ 2\ \bar{1})+q^{4n-13}(1\ \bar{2}\ 0\ 8\ \bar{8}\ \bar{8}\ 21\ \bar{4}\ \overline{17}\ 17\ 2\ \overline{10}\ 6\ 1\ \bar{2}\ 1)+q^{6n-7}(\bar{1}\ 3\ 0\ \bar{8}\ 9\ 8\ \overline{16}\ 3\ 11\ \bar{7}\ \bar{1}\ 3\ \bar{1})+q^{8n+1}(\bar{1}\ 1\ 1\ \bar{3}\ \bar{1}\ 2\ 0\ \bar{1})+q^{10n+9}(\bar{1})$
10_{152}	3		$1^22^3/1^32^2$	$q^{2n}(1)+q^{4n}(0\ 1)+q^{6n}(1\ 0\ 1)+q^{8n}(0\ 1\ 0\ 1)+q^{10n}(1\ \bar{2})$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-3}(1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{8n-5}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{10n-7}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{12n-9}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{14n-11}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{16n-13}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{18n-7}(\bar{1}\ \bar{2}\ \bar{2}\ 0\ \bar{2}\ 5\ \bar{5}\ 0\ \bar{5}\ 7\ \bar{2}\ 4\ \bar{6}\ \bar{2}\ 5\ \bar{2}\ 2\ \bar{2}\ \bar{1})+q^{20n+1}(2\ 3\ 2\ \bar{1}\ 5\ 9\ \bar{1}\ 3\ 9\ 1\ 3\ 4\ 1\ 2)+q^{22n+9}(\bar{1}\ \bar{3}\ 2\ 0\ \bar{5}\ 0\ 1\ \bar{3})$
10_{153}	4		$\bar{1}^223^2/2^23/\overline{12}^2$	$(1\ 0\ 1)+q^{2n}(\bar{1}\ 1)$	$q^{-2n-17}(\bar{1}\ 0\ 0\ \bar{1}\ \bar{1}\ 0\ \bar{1}\ \bar{1}\ \bar{1}\ \bar{1}\ \bar{1}\ 0\ \bar{1})+q^{-17}(1\ 0\ 0\ 1\ 2\ 2\ 0\ 2\ 4\ 0\ 1\ 3\ 1\ 0\ 0\ 1)+q^{2n-7}(\bar{1}\ 2\ 1\ \bar{4}\ 2\ 3\ \bar{3}\ 2\ 2\ \bar{1}\ 0\ 1)+q^{4n-3}(\bar{1}\ 1\ 0\ \bar{3}\ 1\ 1\ \bar{2}\ 0\ 0\ \bar{1})+q^{6n+1}(\bar{1}\ 1\ 0\ \bar{2}\ 1\ 1\ \bar{1})$
10_{154}	4		$1^2\bar{2}3/2^23^2/12^2$	$q^{2n}(1)+q^{4n}(0\ 1)+q^{6n}(1\ 0\ 1)+q^{8n}(\bar{2}\ 1)+q^{10n}(\bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-3}(1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{8n-5}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{10n-7}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{12n-9}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{14n-5}(1\ 0\ \bar{3}\ \bar{1}\ 3\ \bar{2}\ \bar{3}\ 1\ \bar{1}\ 2\ \bar{1}\ 2\ \bar{1})+q^{16n+1}(\bar{2}\ \bar{3}\ 2\ 3\ \bar{5}\ \bar{3}\ 3\ \bar{1}\ \bar{1}\ 0\ \bar{1})+q^{18n+9}(3\ 3\ \bar{4}\ 0\ 4\ 0\ \bar{1})+q^{20n+13}(2\ \bar{1}\ \bar{2}\ 2\ 1)+q^{22n+19}(\bar{1})$
10_{155}	3	10_{137}, $\{1\}su(2)_q$	$1^32/\bar{1}^22/\bar{1}^22$	$q^{2n}(\bar{2}\ 1\ \bar{1})$	$q^{2n-11}(\bar{1}\ 0\ \bar{2}\ 0\ 2\ \bar{5}\ \bar{2}\ 4\ \bar{2}\ \bar{3}\ 1\ 1\ \bar{1})+q^{4n-11}(1\ \bar{1}\ 1\ 2\ \bar{2}\ 4\ 1\ \bar{5}\ 5\ 4\ \bar{5}\ 0\ 4\ \bar{1}\ \bar{1}\ 1)+q^{6n-5}(\bar{1}\ 1\ \bar{1}\ \bar{1}\ 2\ \bar{2}\ \bar{2}\ 3\ \bar{1}\ \bar{3}\ 1\ 1\ \bar{1})$
10_{156}	4	10_{16}, $\{1\}su(n)_q$	$\bar{1}2\bar{3}^2/1^22\bar{3}/1^22$	$q^{2n}(3\ \bar{2}\ 1)$	$q^{2n-13}(\bar{1}\ 2\ 1\ \bar{5}\ 2\ 5\ \bar{5}\ 0\ 4\ \bar{3}\ 0\ 2\ \bar{1})+q^{4n-13}(1\ \bar{2}\ 0\ 7\ \bar{6}\ \bar{8}\ 15\ 0\ \overline{14}\ 10\ 2\ \bar{7}\ 5\ 0\ \bar{2}\ 1)+q^{6n-7}(\bar{1}\ 2\ 1\ \bar{7}\ 3\ 9\ \overline{10}\ \bar{2}\ 8\ \bar{4}\ \bar{1}\ 2\ \bar{1})+q^{8n+3}(\bar{1}\ 1\ 1\ \bar{2}\ 0\ 1)$
10_{157}	3		$\bar{1}^2\bar{2}^3/\bar{1}^22\bar{1}2$	$q^{2n}(1)+q^{4n}(01)+q^{6n}(\bar{3}3\bar{1})$	$q^{-14n-23}(\bar{1}\ 3\ 0\ \bar{8}\ 9\ 3\ \overline{15}\ 7\ 8\ \bar{9}\ 0\ 3\ \bar{1})+q^{-12n-23}(1\ \bar{3}\ 0\ 10\ \overline{14}\ \bar{3}\ 24\ \overline{21}\ \overline{11}\ 28\ \bar{9}\ \overline{14}\ 11\ 1\ \bar{3}\ 1)+q^{-10n-17}(\bar{2}\ 5\ \bar{1}\ \bar{9}\ 12\ 1\ \overline{16}\ 12\ 7\ \overline{13}\ 2\ 6\ \bar{2}\ \bar{1})+q^{-8n-13}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{-6n-9}(1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{-4n-5}(1\ 0\ 0\ 1)+q^{-2n-1}(1)$
10_{158}	4		$1^223/\overline{12}^23/\bar{2}^23$	$(\bar{4}\ 2\ \bar{1})+q^{2n}(\bar{1})$	$q^{-2n-15}(\bar{1}\ 2\ \bar{1}\ \bar{6}\ 7\ 3\ \overline{13}\ 4\ 6\ \bar{6}\ 0\ 2\ \bar{1})+q^{-15}(1\ \bar{2}\ 1\ 6\ \overline{11}\ 1\ 20\ \overline{20}\ \overline{10}\ 23\ \bar{6}\ \bar{9}\ 7\ 0\ \bar{2}\ 1)+q^{2n-9}(\bar{1}\ 3\ \bar{2}\ \bar{5}\ 12\ \bar{3}\ \overline{15}\ 12\ 6\ \bar{9}\ 1\ 3\ \bar{1})+q^{4n-1}(\bar{1}\ 2\ 0\ \bar{4}\ 1\ 2\ \bar{1}\ \bar{1})+q^{6n+7}(\bar{1})$

Knot	BI	Notes	Braid word	$\{1\}su(n)_q$ polynomial	$\{2\}su(n)_q$ polynomial
10_{159}	3		$\bar{1}^2 2^2/1^2\bar{2}/12^2$	$q^{2n}(1)+q^{4n}(3\ \bar{2}\ 1)$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-7}(3\ \bar{3}\ \bar{5}\ 11\ 0\ \overline{12}\ 8\ 4\ \bar{7}\ 2\ 2\ \bar{1})+q^{8n-9}(1\ \bar{2}\ 1\ 4\ \overline{10}\ 4\ 14\ \overline{18}\ \bar{2}\ 17\ \bar{9}\ \bar{5}\ 7\ \bar{1}\ \bar{2}\ 1)+q^{10n-3}(\bar{1}\ 2\ \bar{1}\ \bar{4}\ 7\ \bar{1}\ \bar{9}\ 7\ 3\ \bar{6}\ 1\ 2\ \bar{1})$
10_{160}	4		$1^2 23^2/1^2\bar{2}/1\bar{2}\bar{3}$	$q^{2n}(1)+q^{4n}(\bar{1}\ 2\ \bar{1})+q^{6n}(1)$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-11}(\bar{1}\ 1\ 1\ \bar{2}\ 1\ 2\ \bar{1}\ 0\ 1\ \bar{1}\ 1\ 1\ \bar{1})+q^{8n-11}(1\ \bar{2}\ \bar{2}\ 5\ 0\ \bar{5}\ 3\ 4\ \bar{4}\ 0\ 2\ \bar{2}\ 3\ 0\ \bar{2}\ 1)+q^{10n-5}(\bar{1}\ 1\ 3\ \bar{2}\ \bar{3}\ 5\ 1\ \bar{6}\ 2\ 1\ \bar{1}\ 1\ \bar{1})+q^{12n+3}(1\ \bar{1}\ \bar{1}\ 2\ \bar{1}\ \bar{2}\ 1\ 1)+q^{14n+13}(\bar{1})$
10_{161}	3		$1^2 2^3/1^2 2\bar{1}2$	$q^{2n}(1)+q^{4n}(0\ 1)+q^{6n}(1\ 0\ 1)+q^{8n}(\bar{1}\ 1)$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-3}(1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{8n-5}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{10n-7}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{12n-9}(1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)+q^{14n-5}(1\ 0\ \bar{1}\ 0\ 1\ 0\ \bar{1}\ 0\ 0\ \bar{1}\ 0\ \bar{1})+q^{16n+1}(\bar{1}\ \bar{1}\ 0\ 1\ \bar{1}\ \bar{2}\ 1\ 0\ \bar{1})+q^{18n+11}(1\ \bar{1}\ \bar{1}\ 1\ 0\ \bar{1})$
10_{163}	4		$\bar{1}^2\bar{2}3^2/12^2 3/\bar{2}3$	$q^{2n}(2\ \bar{2})+q^{4n}(1\ \bar{1})$	$q^{2n-9}(\bar{1}\ \bar{3}\ 2\ 0\ \bar{5}\ 1\ 1\ \bar{3})+q^{4n-9}(1\ 2\ \bar{6}\ 0\ 12\ \bar{7}\ \bar{7}\ 10\ \bar{2}\ \bar{4}\ 3)+q^{6n-7}(1\ 1\ \bar{4}\ \bar{2}\ 11\ \bar{1}\ \overline{11}\ 7\ 4\ \bar{5}\ 1\ 1)+q^{8n-3}(1\ \bar{2}\ \bar{2}\ 5\ 0\ \bar{6}\ 2\ 2\ \bar{2})+q^{10n+3}(\bar{1}\ 1\ 1\ \bar{2}\ 0\ 1\ \bar{1})$
10_{164}	4		$1\bar{2}3/\bar{1}^2 2^2/1^2\bar{2}3$	$q^{2n}(4\ \bar{3}\ 1)+q^{4n}(1)$	$q^{2n-13}(\bar{1}\ 3\ 1\ \bar{9}\ 5\ 9\ \overline{12}\ 0\ 9\ \bar{6}\ \bar{1}\ 3\ \bar{1})+q^{4n-13}(1\ \bar{3}\ 0\ 11\ \overline{12}\ \overline{12}\ 30\ \bar{5}\ \overline{26}\ 23\ 3\ \overline{15}\ 8\ 1\ \bar{3}\ 1)+q^{6n-7}(\bar{1}\ 4\ 0\ \overline{13}\ 11\ 13\ \overline{23}\ 1\ 16\ \overline{10}\ \bar{2}\ 4\ \bar{1})+q^{8n+1}(\bar{1}\ 1\ 3\ \bar{2}\ \bar{2}\ 3\ 1\ \bar{1})+q^{10n+9}(\bar{1})$
10_{165}	4		$\bar{1}2\bar{3}^2/1^2 2^2 3/\bar{1}^2 2\bar{3}$	$(\bar{3}\ 2)+q^{2n}(\bar{2}\ 1)$	$q^{-2n-11}(\bar{1}\ \bar{1}\ 6\ \bar{1}\ \bar{8}\ 4\ 3\ \bar{3})+q^{-11}(1\ 0\ \bar{6}\ 10\ 9\ \overline{21}\ 1\ 17\ \bar{6}\ \bar{4}\ 3)+q^{2n-9}(1\ 0\ \bar{7}\ 5\ 13\ \overline{18}\ \bar{8}\ 18\ \bar{1}\ \bar{7}\ 2\ 1)+q^{4n-5}(1\ \bar{3}\ \bar{1}\ 10\ \bar{5}\ \overline{10}\ 9\ 3\ \bar{4})+q^{6n+1}(\bar{1}\ 2\ 1\ \bar{4}\ 1\ 2\ \bar{1})$
10_{166}	4		$1^2 2\bar{3}/2/\bar{1}23^2/\bar{1}2$	$q^{2n}(1)+q^{4n}(3\ \bar{1})+q^{6n}(2\ \bar{1})$	$q^{2n+1}(1)+q^{4n-1}(1\ 0\ 0\ 1)+q^{6n-5}(\bar{1}\ 0\ 5\ \bar{1}\ \bar{6}\ 4\ 3\ \bar{2})+q^{8n-5}(\bar{1}\ \bar{3}\ 2\ 8\ \bar{4}\ \overline{10}\ 6\ 6\ \bar{4}\ \bar{1}\ 1)+q^{10n-3}(1\ \bar{2}\ \bar{3}\ 11\ 1\ \overline{16}\ 4\ 9\ \bar{5}\ \bar{2}\ 1)+q^{12n+1}(1\ \bar{3}\ 2\ 8\ \overline{10}\ 5\ 10\ \bar{1}\ \bar{3}\ 1)+q^{14n+7}(\bar{1}\ 2\ 0\ \bar{4}\ 2\ 2\ \bar{1})$

Chapter 8

Conclusions

In this thesis I have investigated the q-deformed algebras $su(n)_q$ and looked at a major application of these algebras. I have shown that the Racah-Wigner algebra for a q-deformed Lie algebra can be developed in a similar manner to the Racah-Wigner algebra for the corresponding group, however there is an explicit dependence on the deformation parameter q in the vector coupling coefficients and recoupling coefficients. The symmetry properties of the coefficients are complicated by additional q to $1/q$ interchanges. I have proved the q-Biedenharn-Elliott sum rule and the q-Racah backcoupling rule. The former has both q-recoupling coefficients and $\frac{1}{q}$-coefficients while the latter has explicit q-powers. In spite of these differences, the building up method for calculating the coupling and recoupling coefficients can be extended to q-algebras.

The q-deformed algebras have additional structure which the q=1 algebras lack: the presence of non-trivial R-matrices. These matrices effect a $q \rightarrow \frac{1}{q}$ transformation in the coupling coefficients. Together with the coupling coefficients they satisfy the pentagonal equation. I have shown that this equation can be used to recursively calculate R-matrices in an extension of the technique for coupling and recoupling coefficients. Unlike previous methods which require the complete set of vector coupling coefficients to calculate R-matrices, in the present method only the primitive coefficients are required for any R-matrices.

The recursive calculation of coupling coefficients for $su(2)_q$ outlined in this thesis is as straight forward as those known (Hou *et al*, 1990a, 1990b; Kirillov and Reshetikhin, 1988; Groza *et al*, 1990; Ruegg, 1990) giving the same result when differences in the definition of q are taken into account. The derivation in this thesis of the q-$6j$ symbols using the building up method is far simpler and less dependent on the special properties of q-series than those based on Racah's method (Hou *et al*, 1990a, 1990b; Kirillov and Reshetikhin, 1988). The R-matrices for $su(2)_q$ have been calculated recursively which is a more direct route than first calculating q-$6j$ symbols (Nomura, 1989).

I have used the recursive technique to calculate new matrix elements for the R-matrices of $su(3)_q$. As a first step, the algebraic form of the primitive vector coupling coefficients were obtained by using orthogonality relations. Our results agree with those matrix elements and primitive vector coupling coefficients obtained by Ma (1990a, 1990b), namely for the cases of $\{h_1\, h_2\} = \{g_1\, g_2\} = \{1\,0\}$, $\{2\,0\}$ and $\{2\,1\}$. Calculation of the complete form of $su(3)_q$ R-matrices is algebraically involved, but can be obtained in the same manner as the results given here. This work has been published in two papers (Lienert and Butler, 1992a, 1992b).

The R-matrices of q-deformed algebras are used to obtain knot polynomials. The one-variable polynomials based on the $\{1\}$ representation of $su(n)_q$ are a special case of the two-variable HOMFLY polynomial. For knots of ten or fewer crossings and with $n > 2$ the only pairs of knots with the same $\{1\}su(n)_q$ polynomial are those pairs with the same HOMFLY polynomial. Other pairs for the Jones polynomial, which is equivalent to the $\{1\}su(2)_q$ polynomial, are distinguished for higher values of n. I have given the $\{1\}su(n)_q$ polynomials for all knots of ten or fewer crossings.

I have also calculated by recursion the $N = 3$ $\{2\}su(n)_q$ polynomials for all knots of ten or fewer crossings. The recursion has been automated, with the algebraic package MAPLE being used. I have used a similar method to that of Guadagnini (1992) to find two skein-type relations, which together with the Alexander-Conway skein relation are sufficient to determine polynomials for all knots with ten or fewer crossings. However, the properties of R-matrices and coupling coefficients for q-deformed algebras are used rather than appealing to conformal field theory. This work is being prepared for publication.

Akutsu *et al* (1987) show the $\{2\}su(2)_q$ polynomial distinguishes a pair of knots with braid index 3 having the same Jones and HOMFLY polynomials. The $\{2\}su(2)_q$ polynomial was thought to be more powerful than any of the $\{1\}su(n)_q$ polynomials. Extending the calculation both to all the knots of ten or fewer crossings and also to all $\{2\}su(n)_q$ polynomials shows that all pairs for the HOMFLY polynomial of knots of ten or fewer crossings are distinguished by the $\{2\}su(n)_q$ polynomial. However, I have found that the $\{2\}su(n)_q$ polynomial has four new pairs. This is only one fewer than the number of pairs for the $\{1\}su(n)_q$ polynomial for $n > 2$. The $\{2\}su(n)_q$ polynomials do correctly predict nonamphichirality for all knots of ten or fewer crossings, unlike the $\{1\}su(n)_q$ polynomials. I have shown that the $\{1\}$ and $\{2\}$ polynomials together are sufficient to distinguish all knots of ten or fewer crossings.

Acknowledgements

I would like to thank everyone who has been involved in helping me complete this thesis.

Firstly, thankyou to my parents who have supported and encouraged me throughout and learnt a little about 'q-deformed universal enveloping algebras'. My husband Derek has been my computer fix-it person throughout my Ph.D. He also got me started in LaTeX and has helped and supported in a variety of ways.

Secondly, thankyou to Professor Wybourne who started me on the topic of quantum groups as an honours project. Thankyou to my supervisor Assoc. Prof. Phil Butler who has had to learn along with me about q-deformed algebras and has kept me on track. His help in preparing papers and talks has been invaluable.

Thirdly, thankyou to the various members of the department who have taken an interest in my progress and helped in various ways. In particular those who have shared 605 with me through the years: Salam, David Gunn, Huaying Zhang, Nathan Mehrtens, Richard Hugtenberg and Andrew McGregor.

Fourthly, I could not have continued my studies without the financial support received from various groups: a New Zealand Vice Chancellors scholarship, a Ministry of Research, Science and Technology Women's study award and support from the Physics and Astronomy department to attend the 1992 NZIP conference.

Finally, thankyou to my daughter Sarah without whose help this thesis would have been finished somewhat sooner.

References

Abe E 1980 *Hopf Algebras* (*Cambridge Tracts in Mathematics vol 74*) (Cambridge: Cambridge University Press)

Akutsu Y and Wadati M 1987a Knot invariants and critical statistical systems *J. Phys. Soc. Japan* **56** 839–42

——— 1987b Exactly solvable models and new link polynomials I. N-state vertex models *J. Phys. Soc. Japan* **56** 3039–51

Akutsu Y, Deguchi T, and Wadati M 1987 Exactly solvable models and new link polynomials II. Link polynomials for closed 3-braids *J. Phys. Soc. Japan* **56** 3464–79

Alexander J W 1923 A lemma on systems of knotted curves *Proc. Nat. Acad. Sci. USA* **9** 93–95

——— 1928 Topological invariants of knots and links *Trans. Amer. Soc.* **30** 275–306

Alvarez-Gaume L, Gomez C, and Sierra G 1990 Duality and quantum groups *Nucl. Phys. B* **330** 347–88

Andrews G E 1976 *The Theory of Partitions* (*Encyclopedia of Mathematics and its Applications vol 2*) (Reading, MA: Addison-Wesley)

Archer F J 1992 The $U_q sl(3)$ 6-j symbols and state sum model *Phys. Lett. B* **295** 199–208

Arnaudon D and Chakrabarti A 1991 Periodic and partially periodic representations of $SU(N)_q$ *Commun. Math. Phys.* **139** 461–78

Artin E 1925 Theorie der Zöpfe *Abh. Math. Sem. Univ. Hamburg* **4** 47–72

Avancini S S and Menezes D P 1993 Generating function of Clebsch-Gordon coefficients of the $su_q(2)$ quantum algebra *J. Phys. A: Math. Gen.* **26** 1139–46

Baxter R J 1972 Partition function of the eight-vertex lattice model *Ann. Phys.* **70** 193–228

——— 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic Press)

Bérkovich A, Gómez C, and Sierra G 1993 q-magnetism at roots of unity *J. Phys. A: Math. Gen.* **26** L45–52

Biedenharn L C 1989 The quantum group $\mathrm{SU}_q(2)$ and a q-analog of the boson operators *J. Phys. A: Math. Gen.* **22** L873

Bonatsos D, Raychev P P, Roussev R P, and Smirnov Y F 1990 Description of rotational molecular spectra by the quantum algebra $SU_q(2)$ *Chem. Phys. Lett.* **175** 300–306

Broda B 1991 Skein relations for any representation of any Lie group *Phys. Lett. B* **270** 116–122

Burroughs N 1990 The universal R-matrix for $\mathrm{U}_q\mathrm{sl}(3)$ and beyond! *Commun. Math. Phys.* **127** 109–28

Butler P H and Wybourne B G 1976 Calculation of j and jm symbols for arbitrary groups I. Methodology *Int. J. Quantum Chem.* **X** 581–98

Butler P H 1975 Coupling coefficients and tensor operators for chains of groups *Philos. Trans. R. Soc. London* **277** 545–85

——— 1976 Calculation of j and jm symbols for arbitrary groups II. An alternate procedure for angular momentum *Int. J. Quantum Chem.* **X** 599–613

Capps R H 1993 Representations of quantum $su(3)$ *J. Math. Phys.* **34** 5942–50

Celeghini E, Giachetti R, Sorace E, and Tarlini M 1992 Quantum groups of motion and rotational spectra of heavy nuclei *Phys. Lett. B* **280** 180–86

Chang Z and Yan H 1991 The $\mathrm{SU}_q(2)$ quantum group symmetry and diatomic molecules *Phys. Lett. A* **154** 254–8

Conway J 1970 An enumeration of knots and links in *Computational Problems in Abstract Algebra* , edited by Leech J pages 329–358(New York: Pergamon Press)

Couture M 1991 On some quantum R-matrices associated with representations of $U_q(sl(2,\mathbb{C}))$ when q is a root of unity *J. Phys. A: Math. Gen.* **24** L103

Cseh J 1993 ^{16}O+α cluster states in terms of a $U_q(3)$ anharmonic oscillator model *J. Phys. G: Nucl. Phys.* **19** L63–68

Del Sol Mesa A, Loyola G, Moshinsky M, and Velázquez 1993 Quantum groups and the recovery of $U(3)$ symmetry in the Hamiltonian of the nuclear shell model *J. Phys. A: Math. Gen.* **26** 1147–60

Dobrev V K 1990 Characters of the $U_q(sl(3,\mathbb{C}))$ highest weight modules *Progress of Theoretical Physics Supplement* **102** 137–158 Common Trends in Mathematics and Quantum Field Theories, 1990 Yukawa International Seminar

Drinfel'd V G 1985 Hopf algebras and the quantum Yang-Baxter equation *Sov. Math.-Dokl.* **32** 254

——— 1986 Quantum groups in *Proceedings of the International Congress of Mathematics* pages 798–820(Berkeley, California: American Mathematical Society)

Edmonds A R 1957 *Angular Momentum in Quantum Mechanics* (*Investigations in physics vol 4*) (Princeton, New Jersey: Princeton University Press)

Freyd P, Yetter D, Hoste J, Lickorish W B R, Millet K, and Ocneanu O 1985 A new polynomial invariant of knots and links *Bull. AMS* **12** 183–312

Fu H C and Ge M L 1992 The q-boson realization of parametrized cyclic representations of quantum algebras at $q^p = 1$ *J. Math. Phys.* **25** 427–35

Gómez C, Ruiz-Altaba M, and Sierra G 1991 New R-matrices associated with finite dimensional representations of $U_q(sl(2))$ at roots of unity *Phys. Lett. B* **265** 95–98

Groza V A, Kachurk I I, and Klimyk A U 1990 On Clebsch-Gordan coefficients and matrix elements of representations of the quantum algebra $\mathrm{U}_q(\mathrm{su}_2)$ *J. Math. Phys.* **31** 2769–80

Guadagnini E 1992 The universal link polynomial *Int. J. Mod. Phys. A* **7** 877–946

Hakobyan T S and Sedrakyan A G 1993 R-matrices for highest weight representation of $\widehat{sl_q(2,C)}$ at roots of unity *Phys. Lett. B* **303** 27–33

Hou B Y, Hou B Y, and Ma Z Q 1990a Clebsch-Gordan coefficients, Racah coefficients and braiding fusion of quantum sl(2) enveloping algebra I *Commun. Theor. Phys.* **13** 181–98

——— 1990b Clebsch-Gordan coefficients, Racah coefficients and braiding fusion of quantum sl(2) enveloping algebra II *Commun. Theor. Phys.* **13** 341–54

Iwao S 1990a Knot and conformal field theory approach in molecular and nuclear physics *Prog. Theor. Phys.* **83** 363–67

——— 1990b Spectroscopy in Witten's quantum universal enveloping algebra of su_2 and strength of deformation of its representation space *Prog. Theor. Phys.* **83** 368

Jimbo M, Misra K C, Miwa T, and Okado M 1991 Combinatorics of representations of $U_q(\widehat{sl}(n))$ at q=0 *Commun. Math. Phys.* **136** 543–66

Jimbo M 1985 A q-difference analogue of $U(\mathcal{G})$ and the Yang-Baxter equation *Lett. Math. Phys.* **10** 63–9

Jones V F R 1985 A polynomial invariant for knots via von Neumann algebras *Bull. AMS* **12** 103–111

——— 1987 Hecke algebra representations of braid groups and link polynomials *Ann. Math.* **126** 335–88

Kachurik I I and Klimyk A U 1990 On Racah coefficients of the quantum algebra $U_q(su_2)$ *J. Phys. A: Math. Gen.* **23** 2717–28

——— 1991 General recurrence relations for Clebsch-Gordan coefficients of the quantum algebra $U_q(su_2)$ *J. Phys. A: Math. Gen.* **24** 4009–15

Kauffman L H 1990 An invariant of regular isotopy *Trans. Amer. Math. Soc.* **318** 417–471

——— 1991 *Knots and Physics* (*Series on Knots and Everything vol 1*) (Singapore: World Scientific)

Kirillov A N and Reshetikhin N Y 1988 Representations of the algebra $U_q(sl_2)$, q-orthogonal polynomials and invariants of links in *Infinite Dimensional Lie Algebras and Groups* , edited by Kac V G pages 285–339(World Scientific) LOMI E-9-88

Koelink H T and Koornwinder T H 1989 The Clebsch-Gordan coefficients for the quantum group $S_\mu U(2)$ and q-Hahn polynomials *Proc. Kon. Ned. Akad. van Wetensch. A* **92** 443–56

Kulish P P and Sklyanin K 1980 Quantum version of the method of inverse scattering problem *Sov. Math.-Dokl.* **19** 1547

Lienert C R and Butler P H 1992a Racah-Wigner algebra for q-deformed algebras *J. Phys. A: Math. Gen.* **25** 1223–1235

——— 1992b Recursive calculation of the R-matrices of q-deformed algebras *J. Phys. A: Math. Gen.* **25** 5577–86

Lusztig G 1988 Quantum-deformations of certain simple modules over enveloping algebras *Adv. Math.* **70** 237–49

Ma Z Q 1990a Representations of the braid group obtained from quantum sl(3) enveloping algebra *J. Math. Phys.* **31** 550–56

——— 1990b New link polynomial obtained from octet representation of quantum sl(3) enveloping algebra *J. Math. Phys.* **31** 3079–84

Macfarlane A J 1989 On q-analogues of the harmonic oscillator and the quantum group $SU_q(2)$ *J. Phys. A: Math. Gen.* **22** 4581–88

Manin Y I 1987 Some remarks on Koszul algebras and quantum groups *Ann. Inst. Fourier (Grenoble)* **37** 191–205

Maple V ©1981-90 Waterloo Maple Software, University of Waterloo

Markov A A 1945 Foundations of the algebraic theory of braids *Trudy Math. Inst. Steklov* **16** 1–54

Matsuda T, Mimachi K, Nakagami Y, et al. 1988 Representations of quantum groups and a q-analog of orthogonal polynomials *C. R. Acad. Sci. Paris* **307** 559

Moshinsky M 1962 Wigner coefficients for the SU_3 group and some applications *Rev. Mod. Phys.* **34** 813–28

Nomura M 1989 Relations for Clebsch-Gordan and Racah coefficients in $SU_q(2)$ and Yang-Baxter algebras *J. Math. Phys.* **30** 2397–405

——— 1990 Representation functions d^j_{mk} of $U[sl_q(2)]$ as wave functions of 'quantum symmetric tops' and relationship to braiding matrices *J. Phys. Soc. Japan* **59** 4260–71

Pan F and Chen J Q 1992 Symmetric irreducible representations of $SU(3)_q$ in the $SU(2)_q \times U(1)$ basis and some extensions to $SU(N)_q SU(N-1)_q \times U(1)$ *J. Phys. A: Math. Gen.* **25** 4017–24

Pan F 1993 Quantum deformation of $SU(3)$ and subalgebras *J. Phys. A: Math. Gen.* **26** L257–62

Quesne C 1992 Complementarity of $su_q(3)$ and $u_q(2)$ and q-boson realization of the $su_q(3)$ irreducible representations *J. Phys. A: Math. Gen.* **25** 5977–5998

Racah G 1942 Theory of complex spectra. II *Phys. Rev.* **62** 438–462 Reprinted in Quantum Theory of Angular Momentum, Ed. LC Biedenharn and H Van Dam

Raychev P P, Roussev R P, and Smirnov Y F 1990 The quantum algebra $\mathrm{SU}_q(2)$ and rotational spectra of deformed nuclei *J. Phys. G: Nucl. Phys.* **16** L137–142

Reshetikhin N Y 1987 Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links *Preprint* LOMI St. Petersburg E-4-87 and E-17-87

Rolfsen D 1976 *Knots and Links* (Berkeley: Publish or Perish)

Rosso M 1988 Finite dimensional representation of the quantum analog of the enveloping algebra of a complex simple Lie algebra *Commun. Math. Phys.* **117** 581–94

Ruegg H 1990 A simple derivation of the quantum Clebsch-Gordan coefficients for $\mathrm{su}(2)_q$ *J. Math. Phys.* **31** 1085–87

Ruiz-Altaba M 1992 New solutions to the Yang-Baxter equation from two-dimensional representations of $U_q(sl(2))$ at roots of unity *Phys. Lett. B* **279** 326–332

Sklyanin E K, Takhtajan L A, and Faddeev L D 1979 Quantum inverse problem method I *Theor. Math. Phys.* **40** 194

Sklyanin E K 1982 Some algebraic structures connected with the Yang-Baxter equation *Funkt. Anal. Pril.* **16** 27–34 (Engl. Transl. 1982 Funct. Anal. Appl. **16** 263)

Sun C P and Ge M L 1991a The q-deformed boson realization of representation of quantum universal enveloping algebras for q a root of unity *J. Phys. A: Math. Gen.* **24** 3265–80

——— 1991b q-analogues of Verma representations of quantum algebras for q a root of unity: The case of $U_q(sl(3))$ *J. Phys. A: Math. Gen.* **24** 3731–3740

——— 1991c The cyclic boson operators and new representations of the quantum algebras $sl_q(2)$ for q a root of unity *J. Phys. A: Math. Gen.* **24** L969–74

Sun C P and Ge M L 1992 The q-deformed boson realization of representations of quantum universal enveloping algebras for q a root of unity: II The subalgebra chains of $U_q(C_l)$) *J. Phys. A: Math. Gen.* **25** 401–10

Sun C P, Lu J F, and Ge M L 1990 Structure of regular representations of quantum enveloping algebras $\mathrm{SL}_q(2)$ with $q^p = 1^*$ *J. Phys. A: Math. Gen.* **23** L1199

Tait P G 1898 *On Knots I, II, III.* (*Scientific papers vol 1*) (London: Cambridge University Press)

Vaksman L L 1989 The q-analogs of the Clebsch-Gordan coefficients and algebra of functions on quantum group $SU(2)$ *Sov. Math.-Dokl.* **306** 269–71 Originally published in Dokl. Akad. Nauk. SSSR, **288** 1988

Wigner E P 1931 *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Braûnschweig: Vieweg)

Woronowicz S L 1987a Twisted $SU(N)$ group. an example of a non-commutative differential calculus *Publ. RIMS (Kyoto University)* **23** 117–181

——— 1987b Compact matrix pseudogroups *Commun. Math. Phys.* **111** 613–665

Yang C N 1967 Some exact results for the many-body problem in one dimension with repulsive delta-function interaction *Phys. Rev. Lett.* **19** 1312–16

Yu Z 1991 A new method of determining an irreducible representation of quantum $Sl_q(3)$ algebra (I) *J. Phys. A: Math. Gen.* **24** L399–402

Zhang R B, Gould M D, and Bracken A J 1991 Quantum group invariants and link polynomials *Commun. Math. Phys.* **137** 13–28

Zhang Y J 1992 A remark on the representation theory of the algebra $U_q(sl(n))$ when q is a root of unity *J. Phys. A: Math. Gen.* **25** 851–60